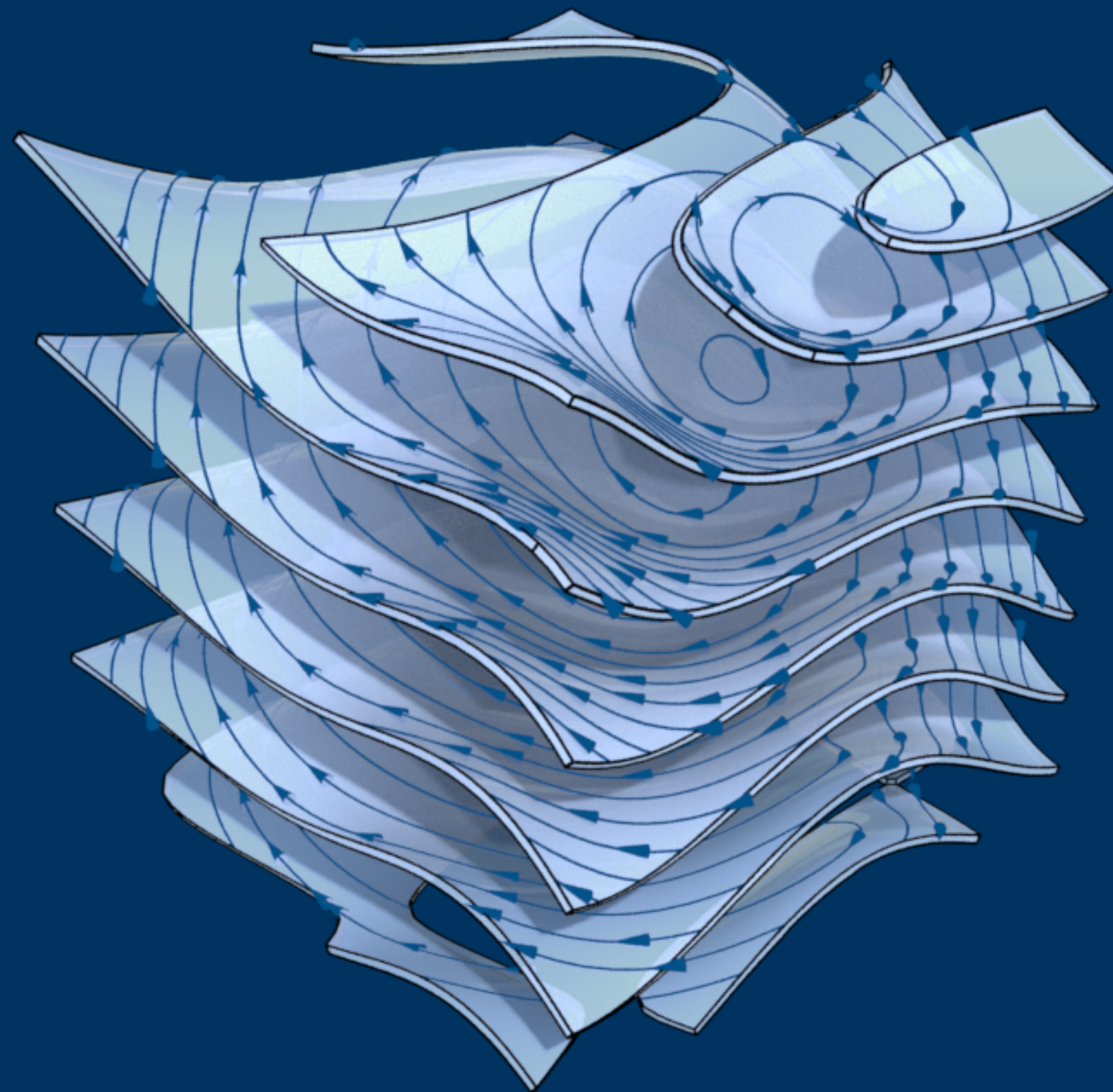
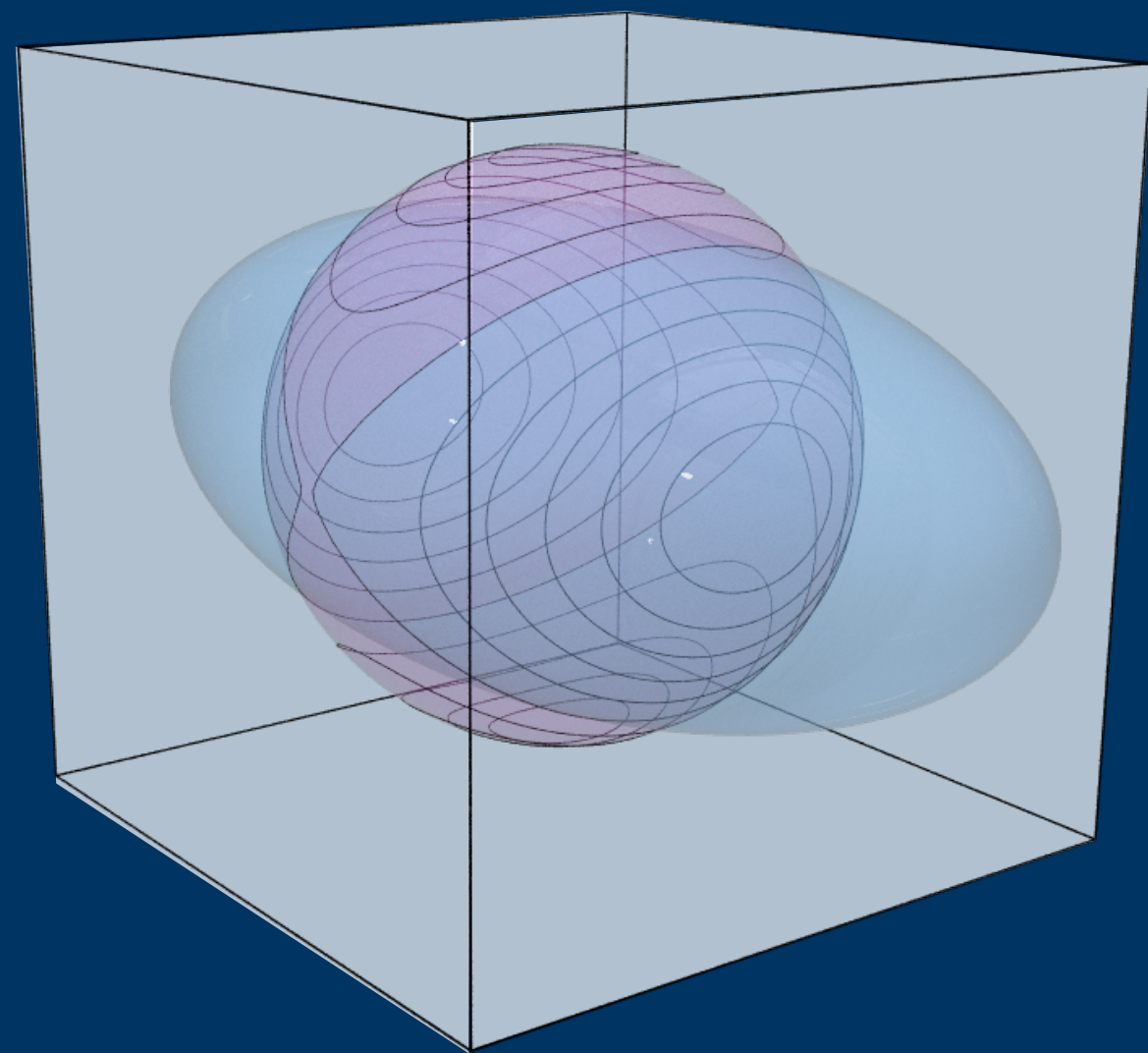




June 29th, 2026

SGP 2026 Graduate School

Introduction to Geometric Fluid Mechanics

Mohammad Sina Nabizadeh

PostDoc @ MIT

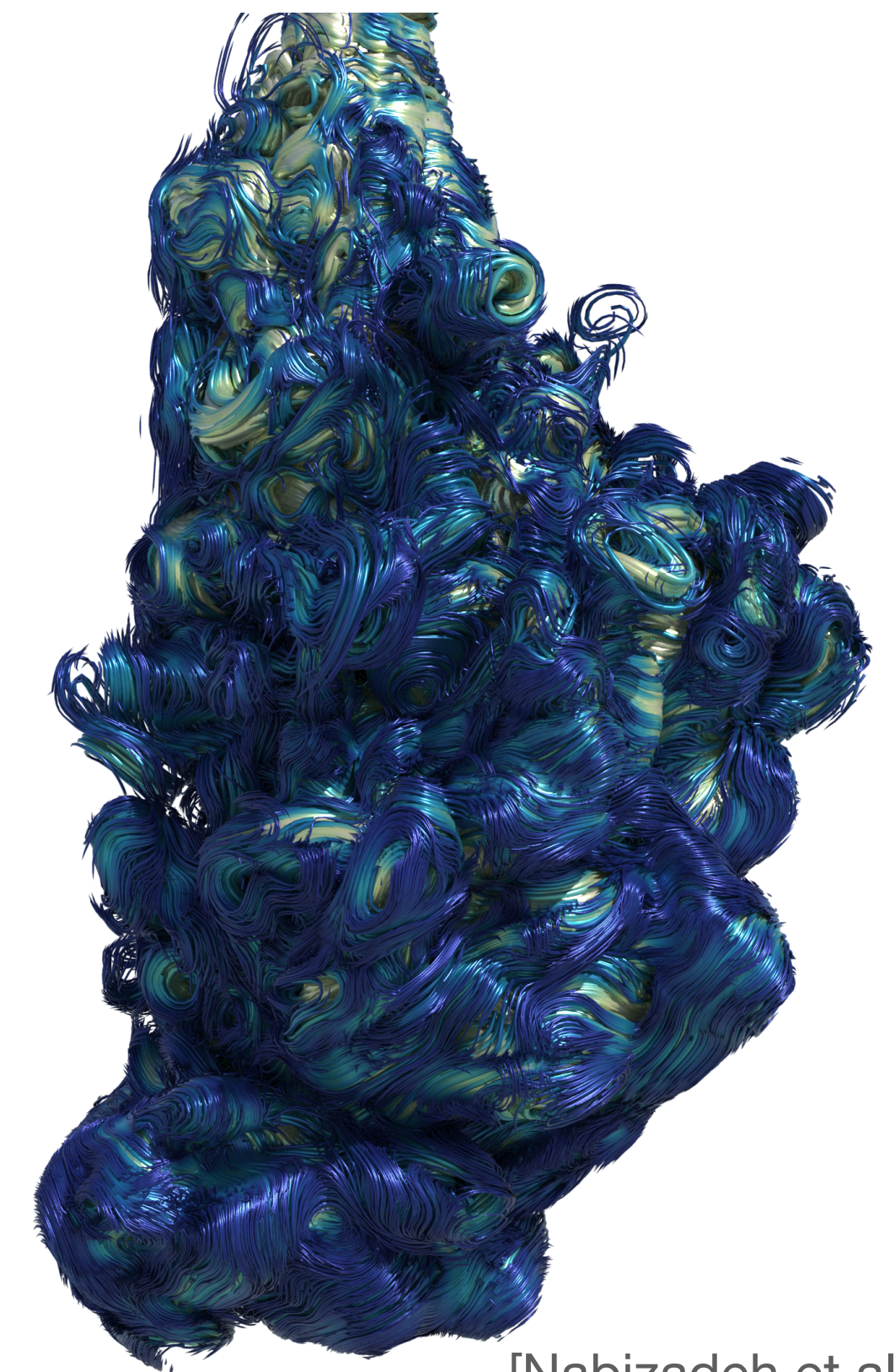
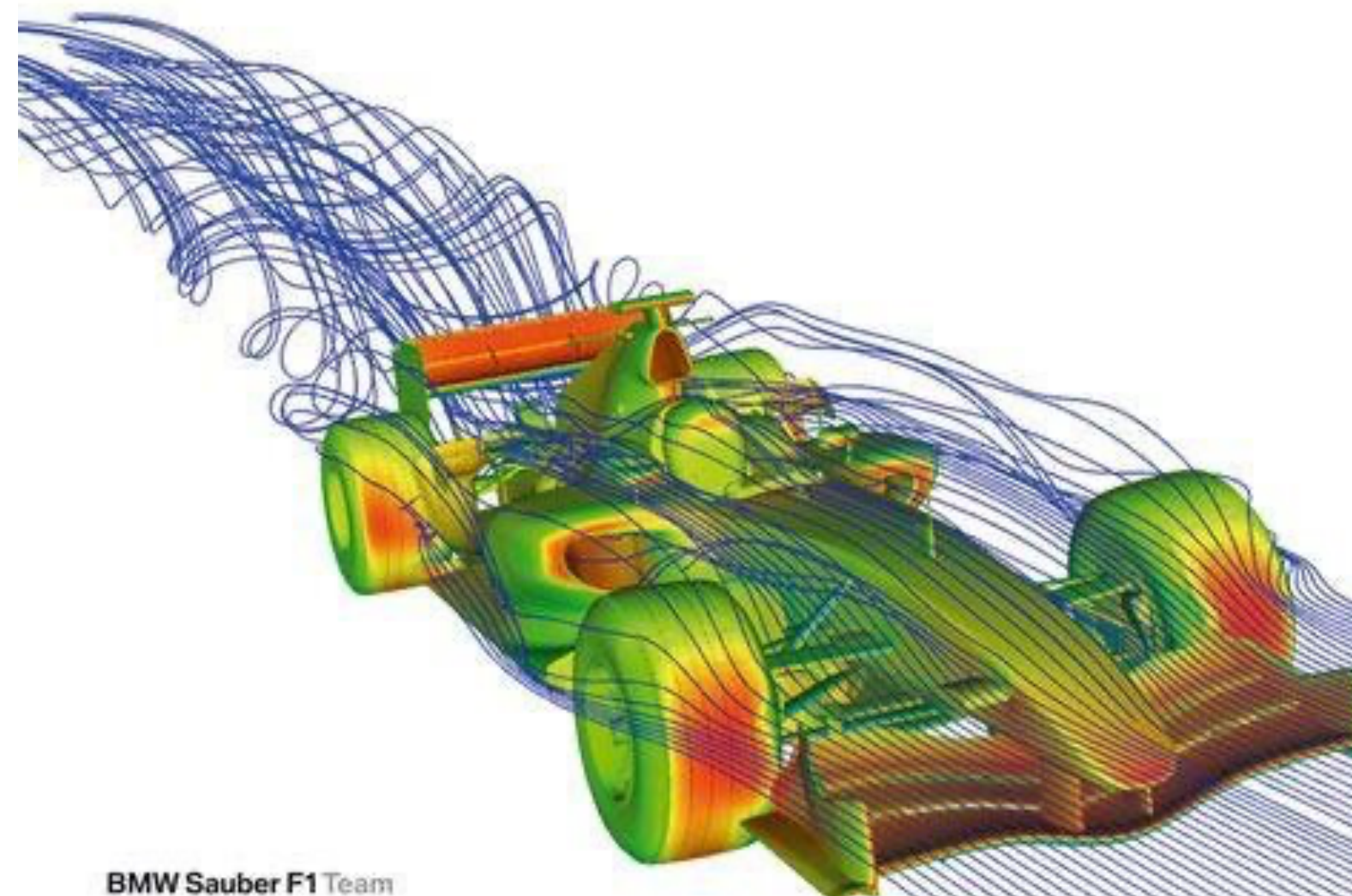
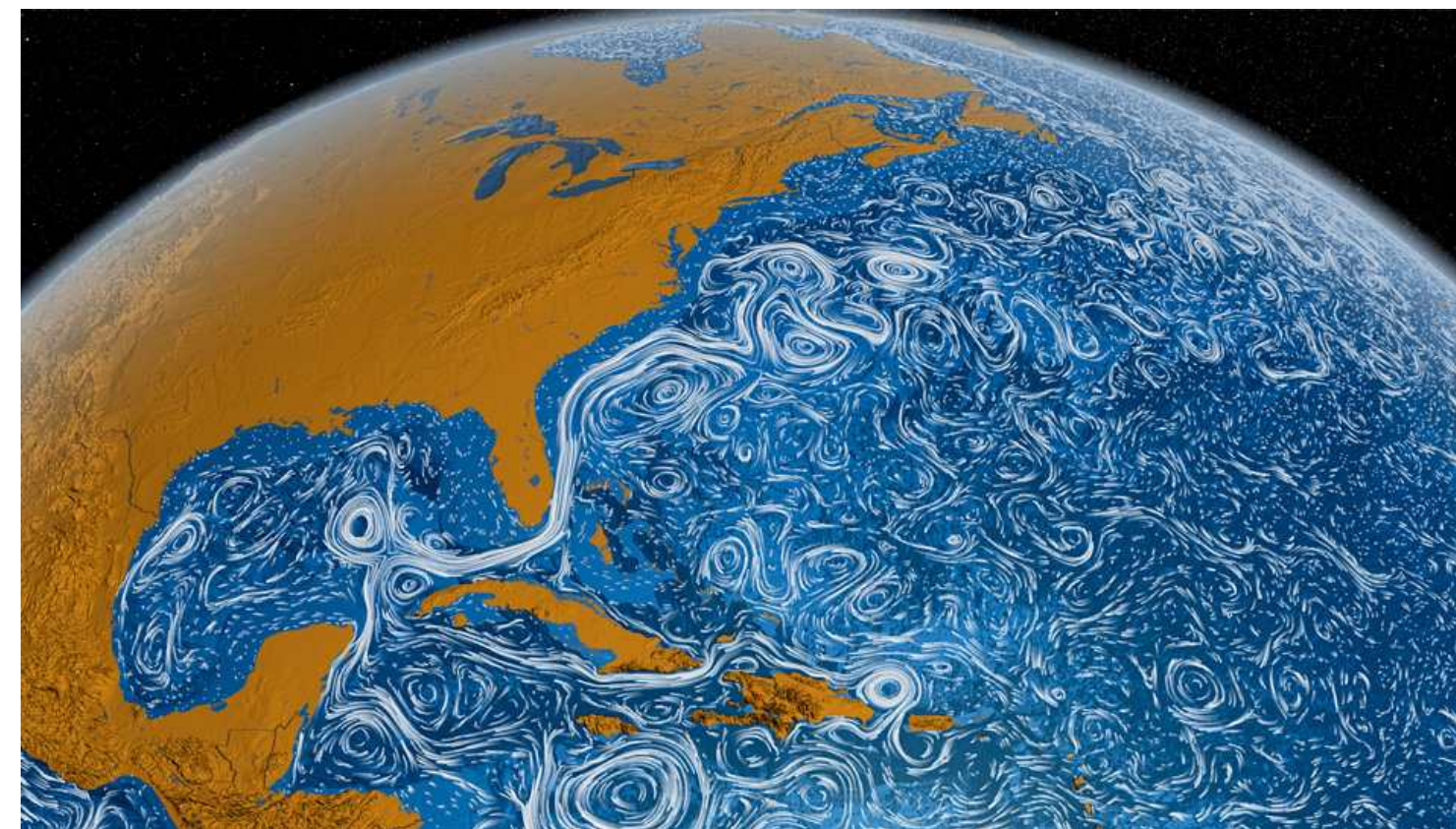
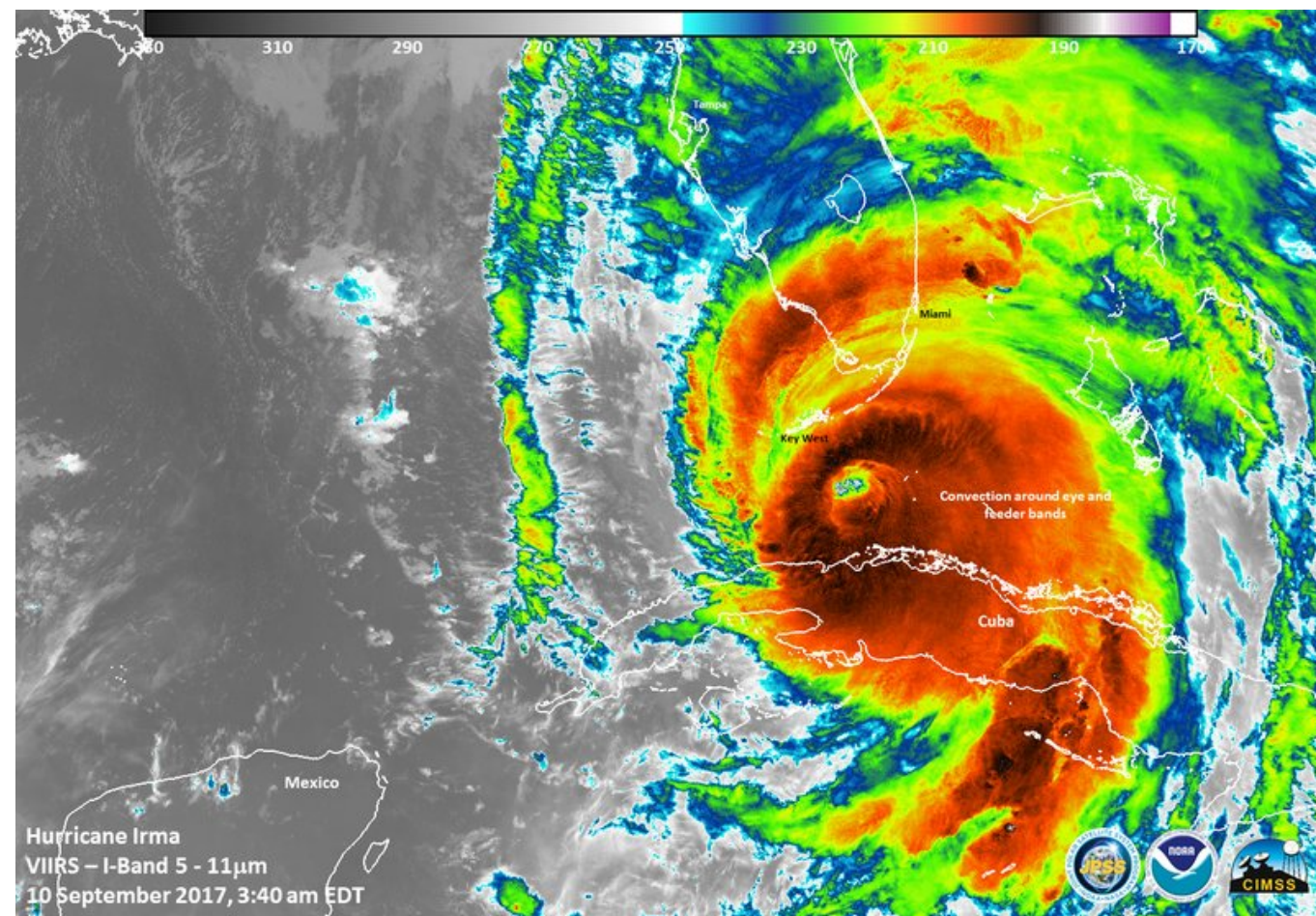
Hang “Hesper” Yin

PhD Candidate @ UCSD

Introduction

Why should geometric mechanics be a part of SGP?

Fluid simulations impact us everyday



[Nabizadeh et al. 2022]

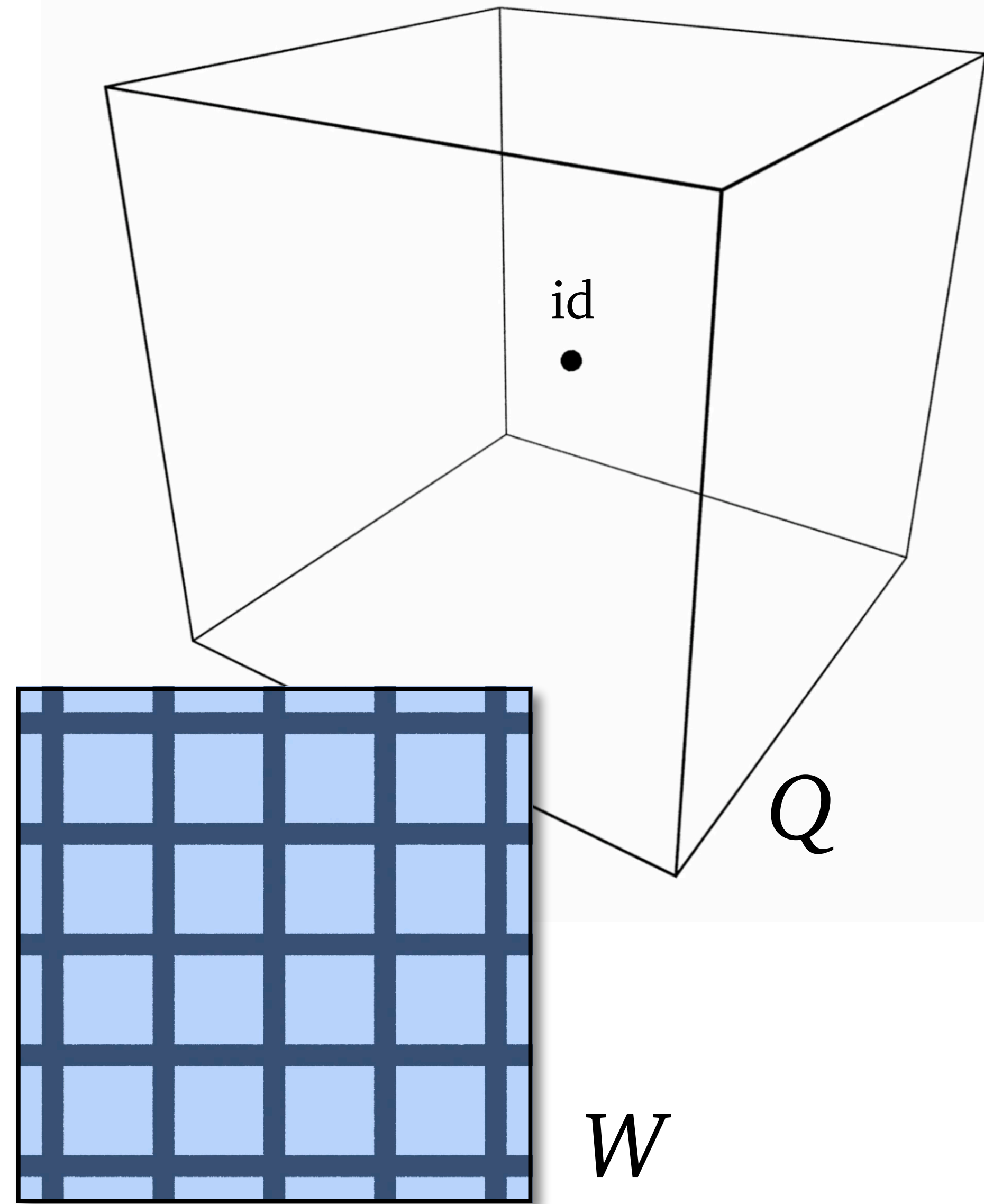
Why should geometric mechanics be a part of SGP?

Euler's equations

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

Navier-Stokes' equations

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$



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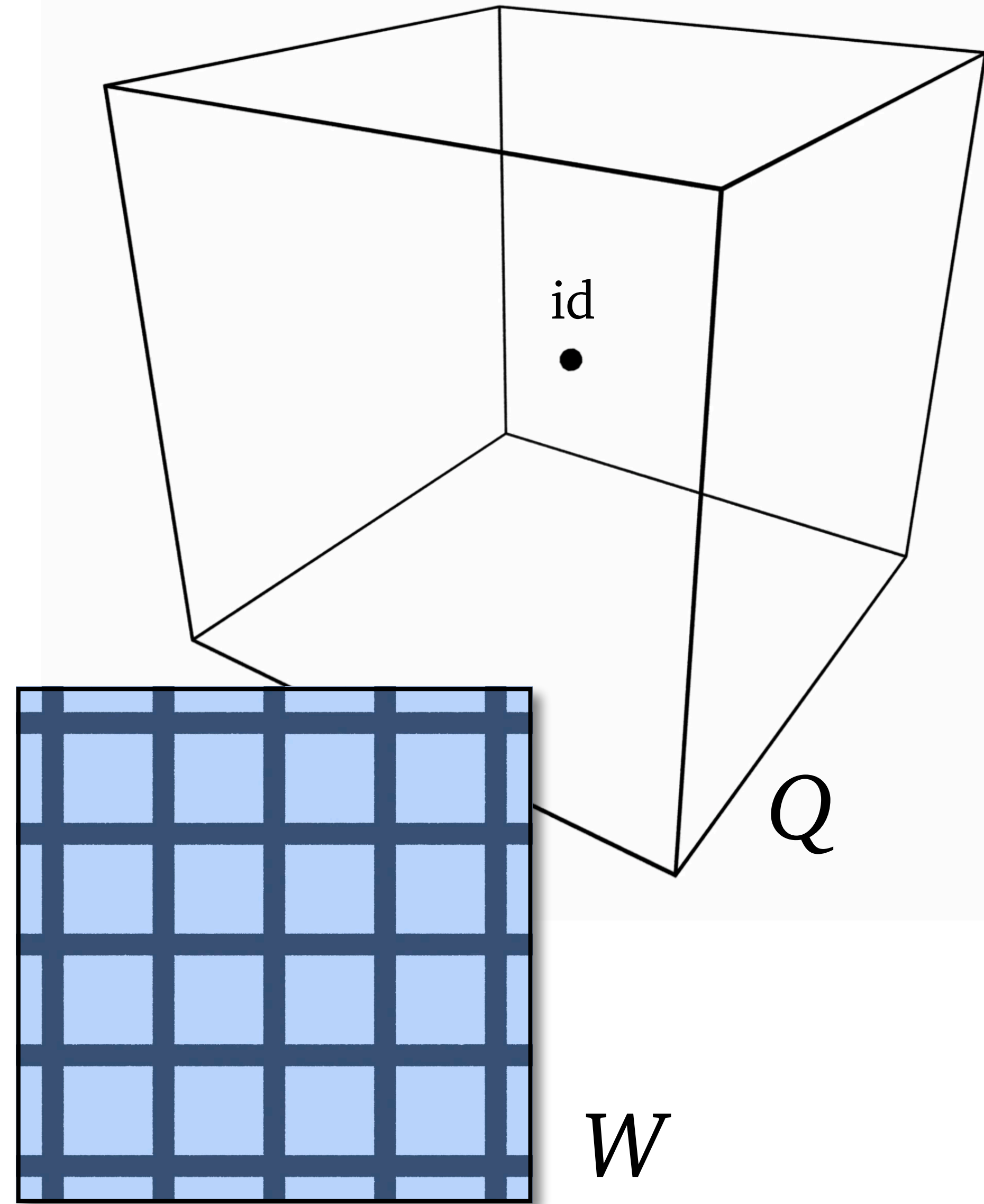
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Hyperbolic PDEs



Why should geometric mechanics be a part of SGP?

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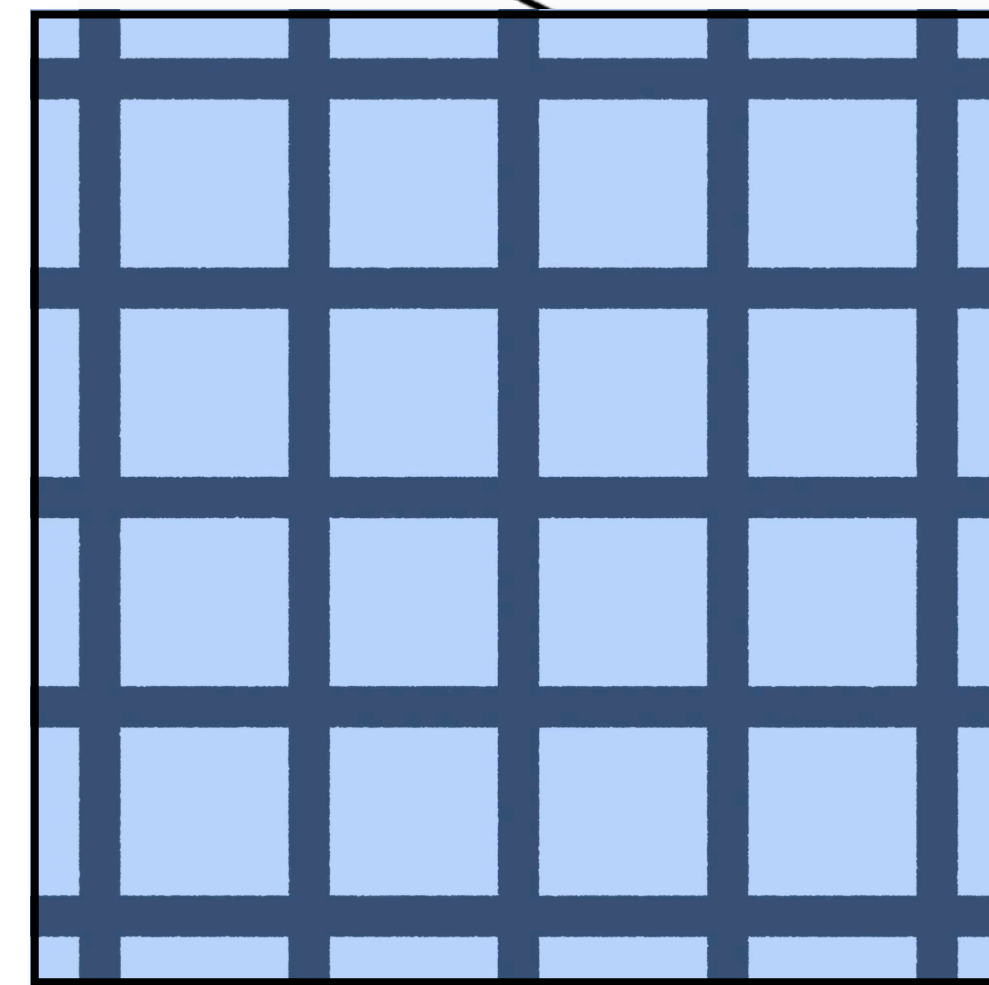
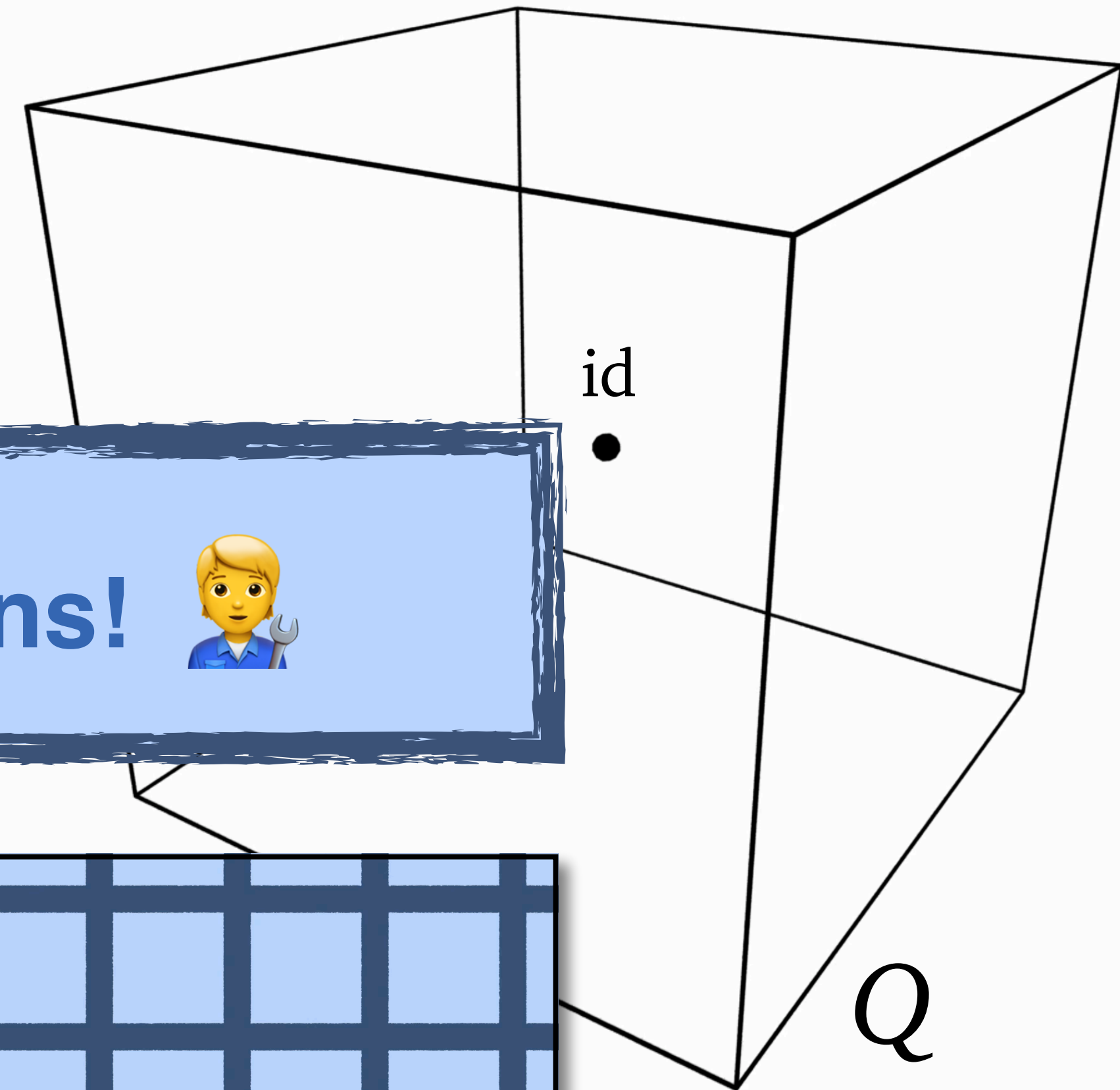
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Hyperbolic PDEs

Structure-preserving simulations! 🧑🔧



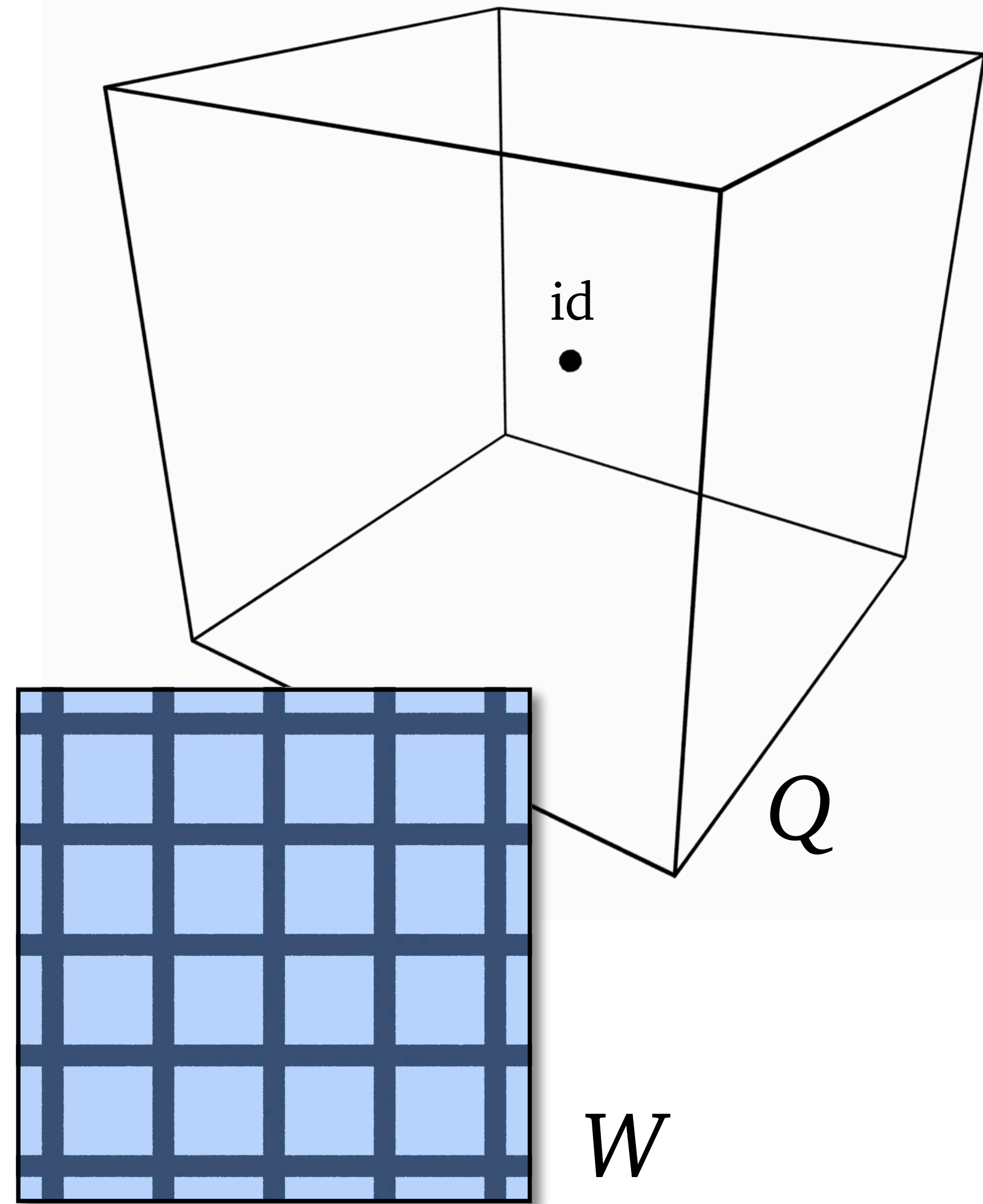
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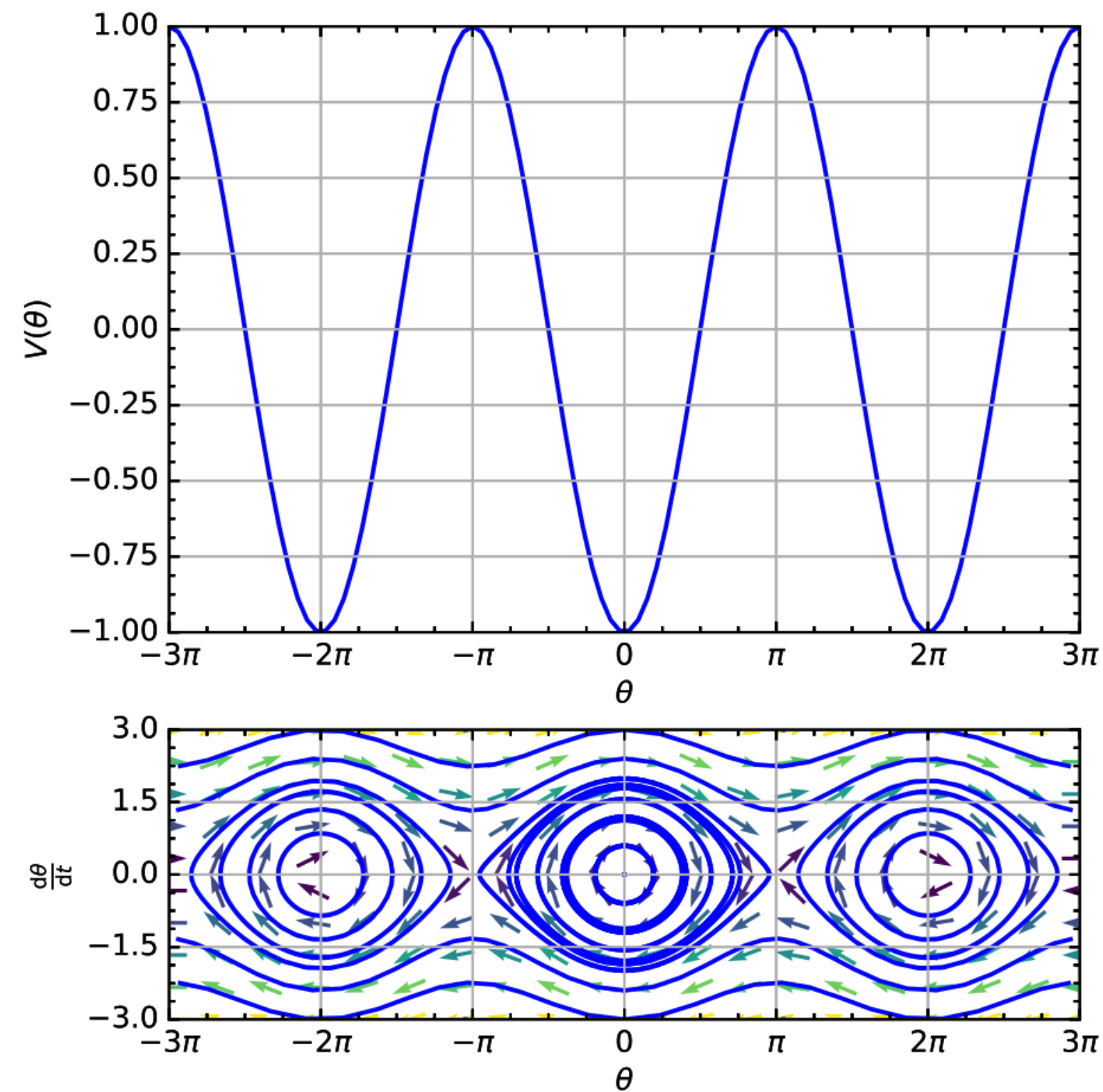
Why should geometric mechanics be a part of SGP?

Our mathematical tools:

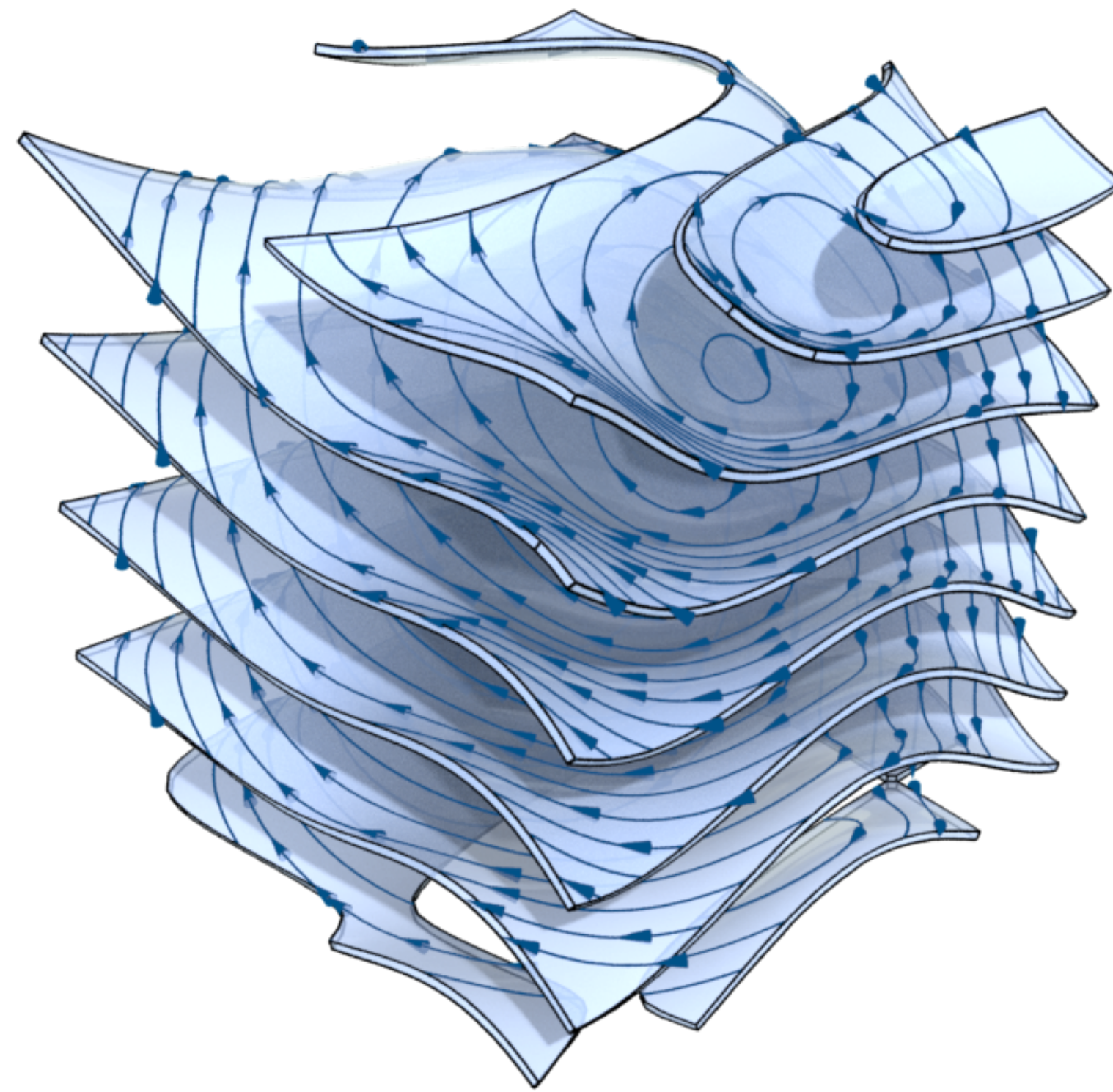
- Variational Optimization
 - ↳ Lagrangian Mechanics
 - Symplectic Geometry
 - ↳ Hamiltonian Mechanics
 - Lie Groups
 - ↳ Symmetry
- and Conservation Laws



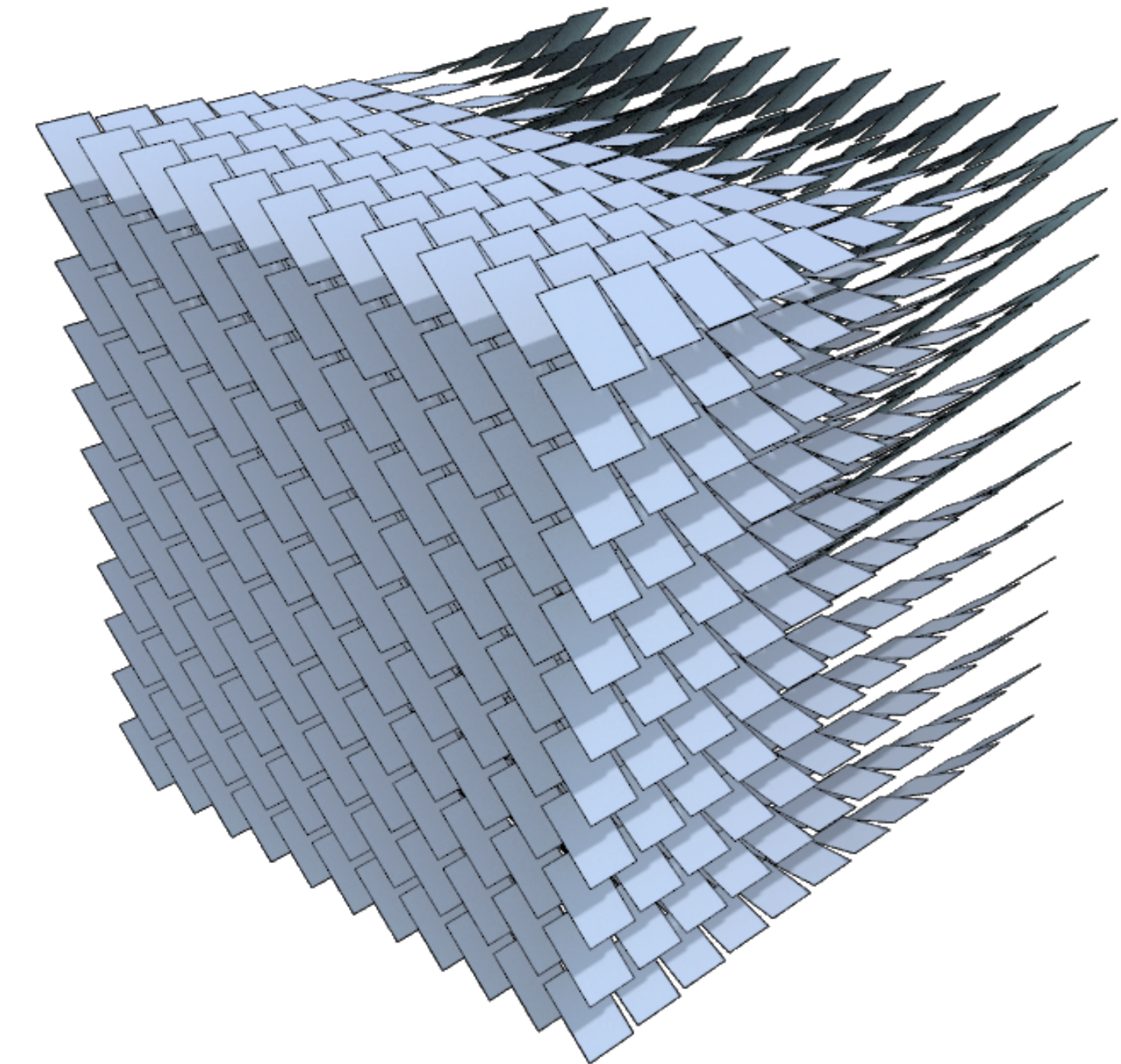
Overview



**Lagrangian and
Hamiltonian Mechanics**

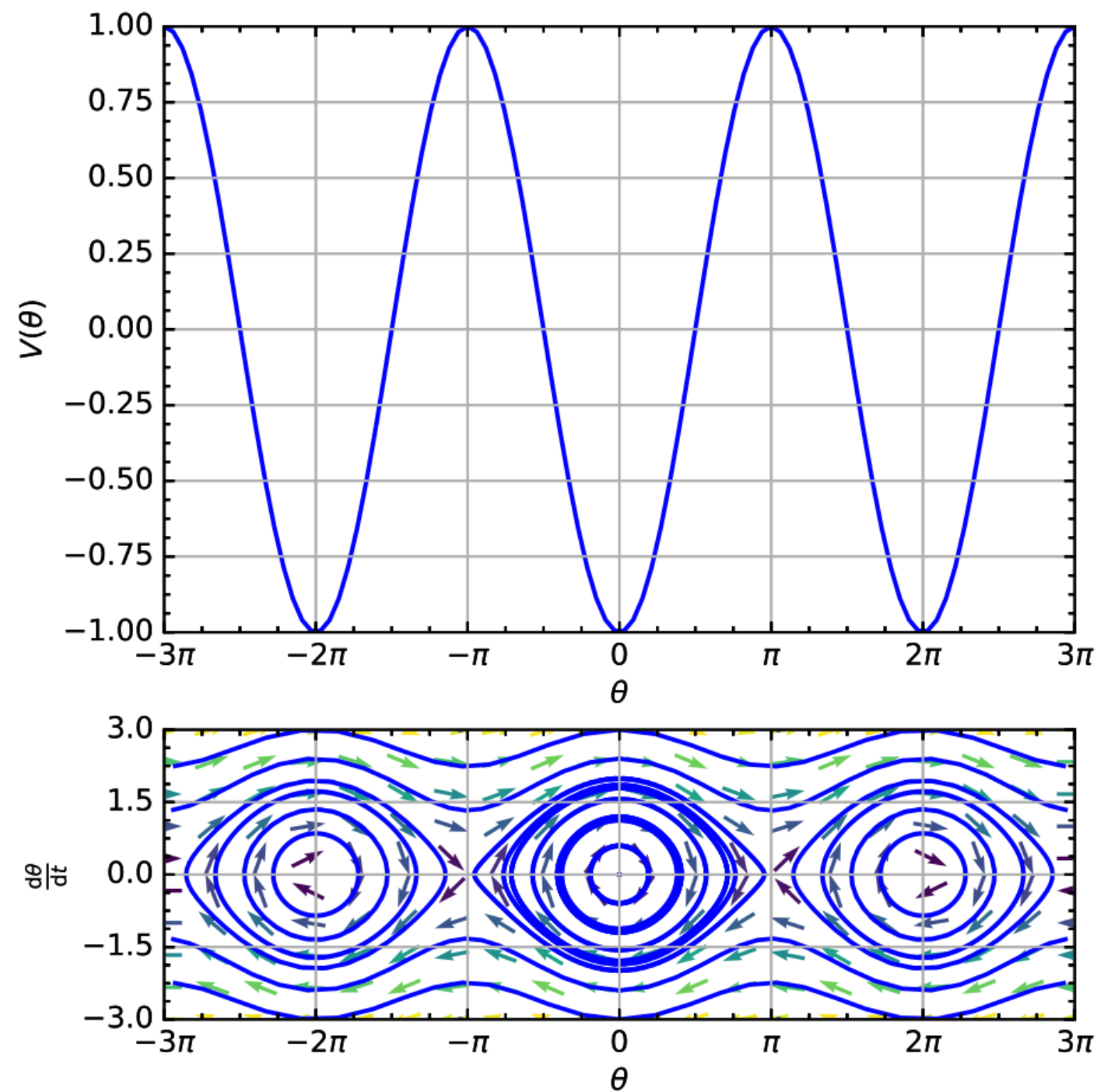


**Lie groups, Symmetry,
and Reduction**

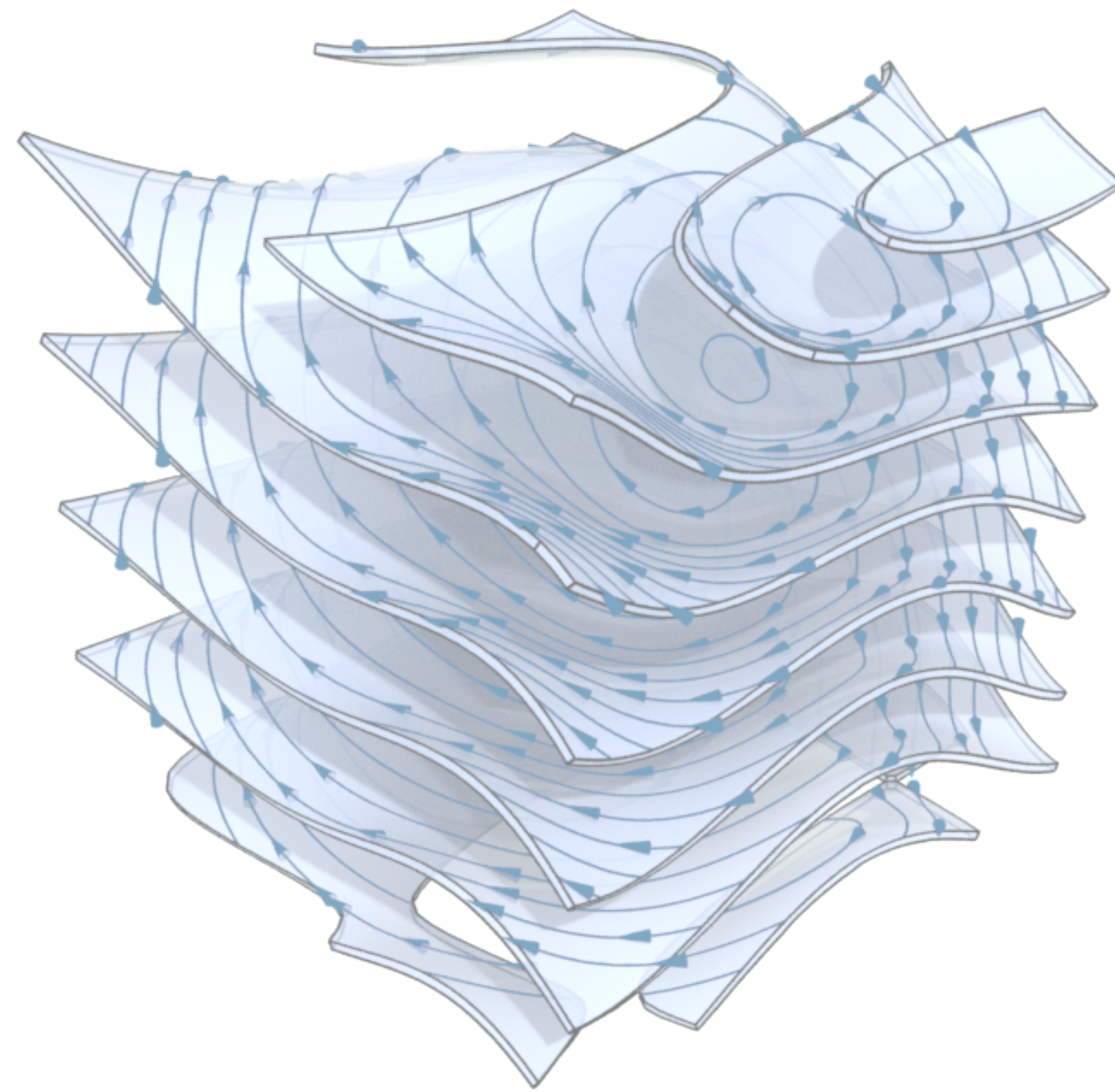


**Discretization as a
constrained variational
problem**

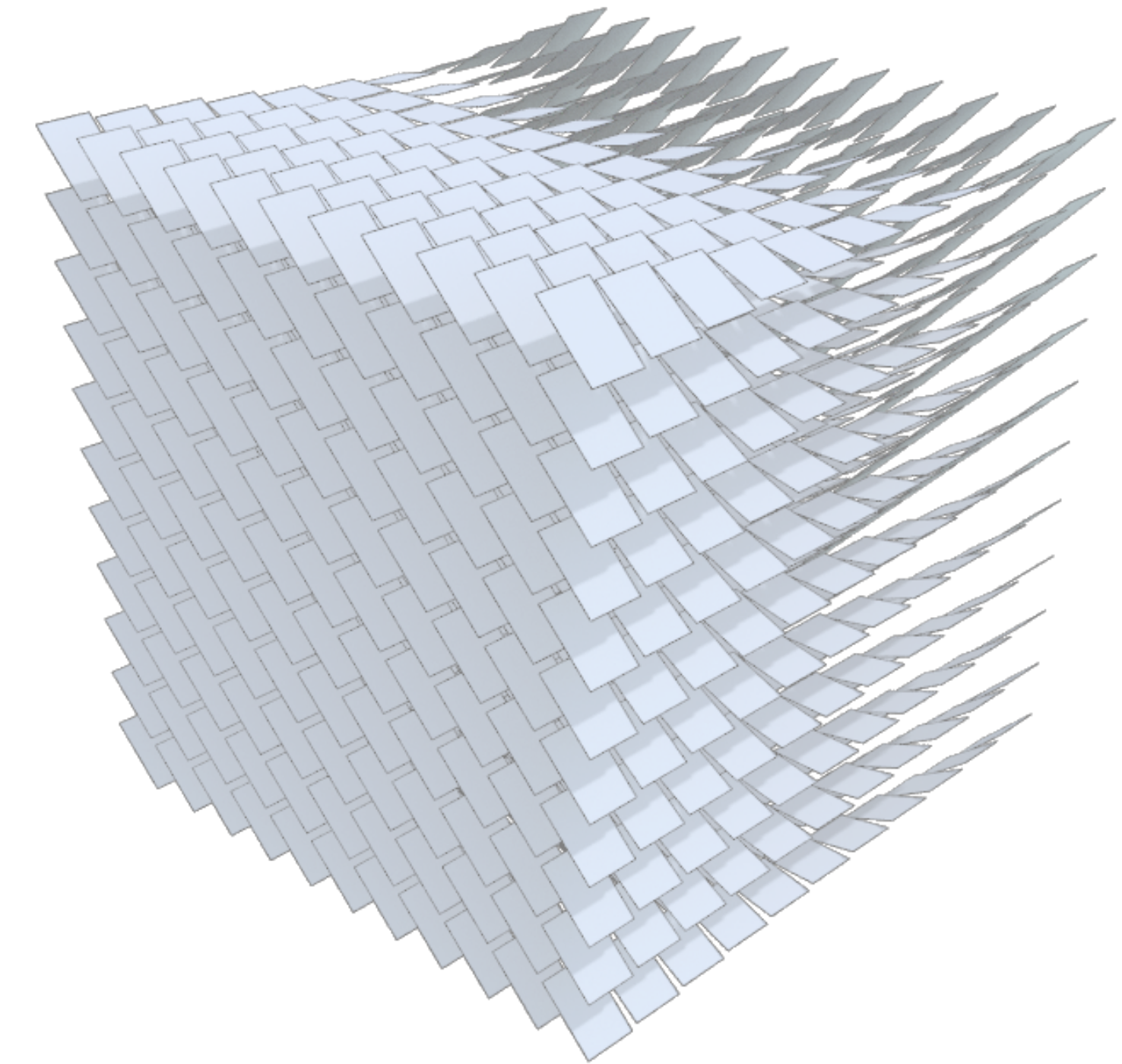
Overview



**Lagrangian and
Hamiltonian Mechanics**



**Lie groups, Symmetry,
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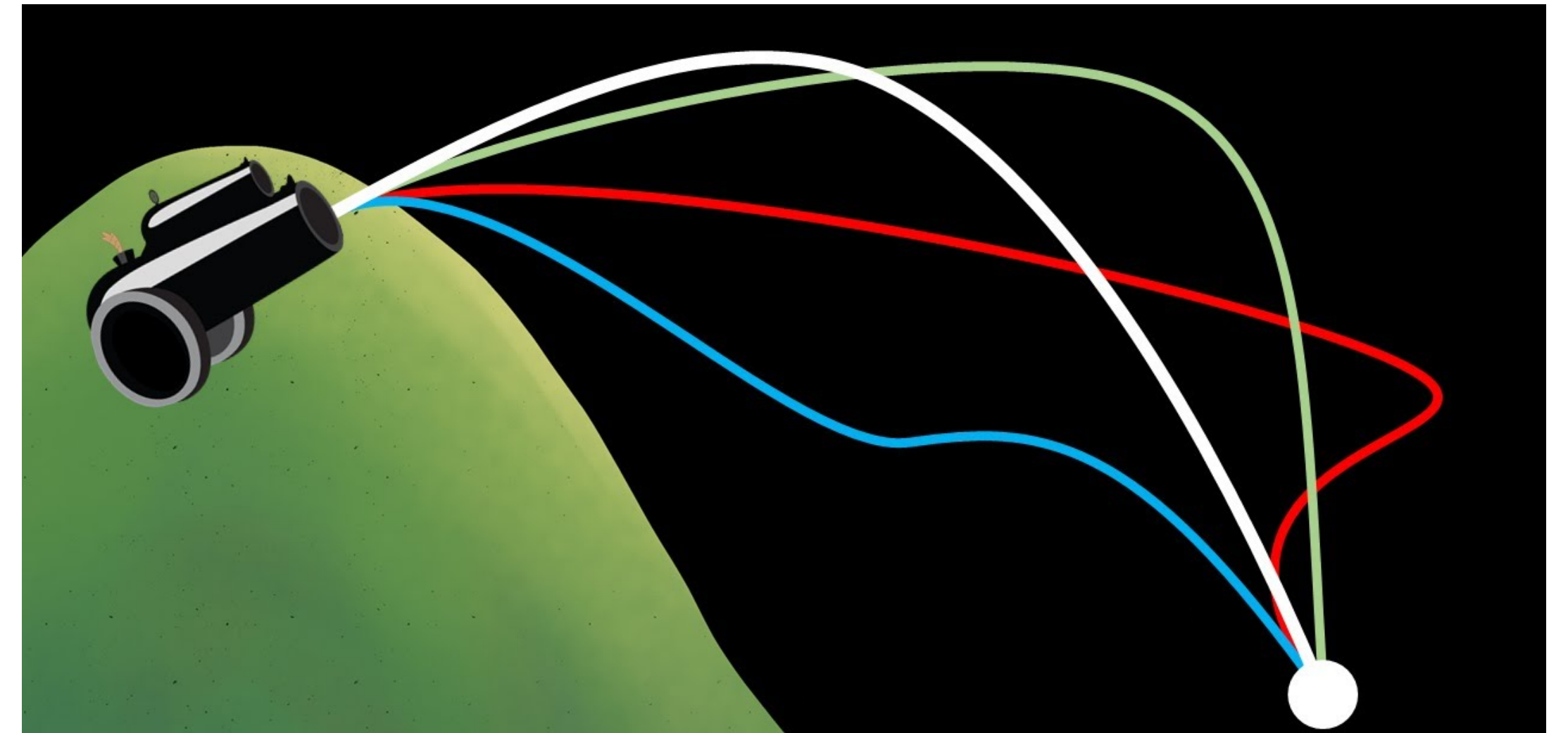
**Discretization as a
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Physical world

An optimization problem

Physical world

An optimization problem

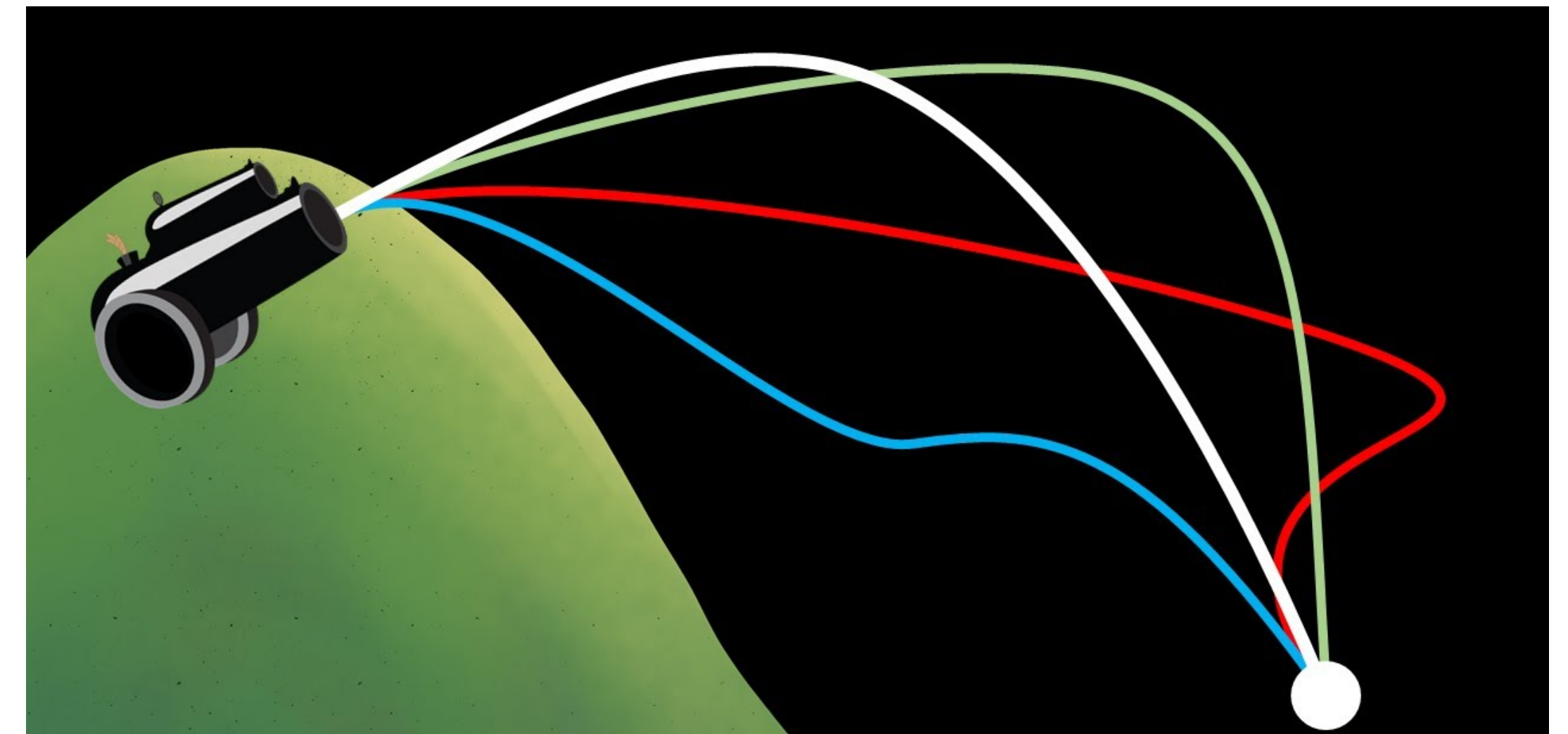


[YT Dr Mitchell's physics channel 2020]

Physical world

An optimization problem

- Least action principle
- The physical motion follows the path of least total cost (action)



[YT Dr Mitchell's physics channel 2020]

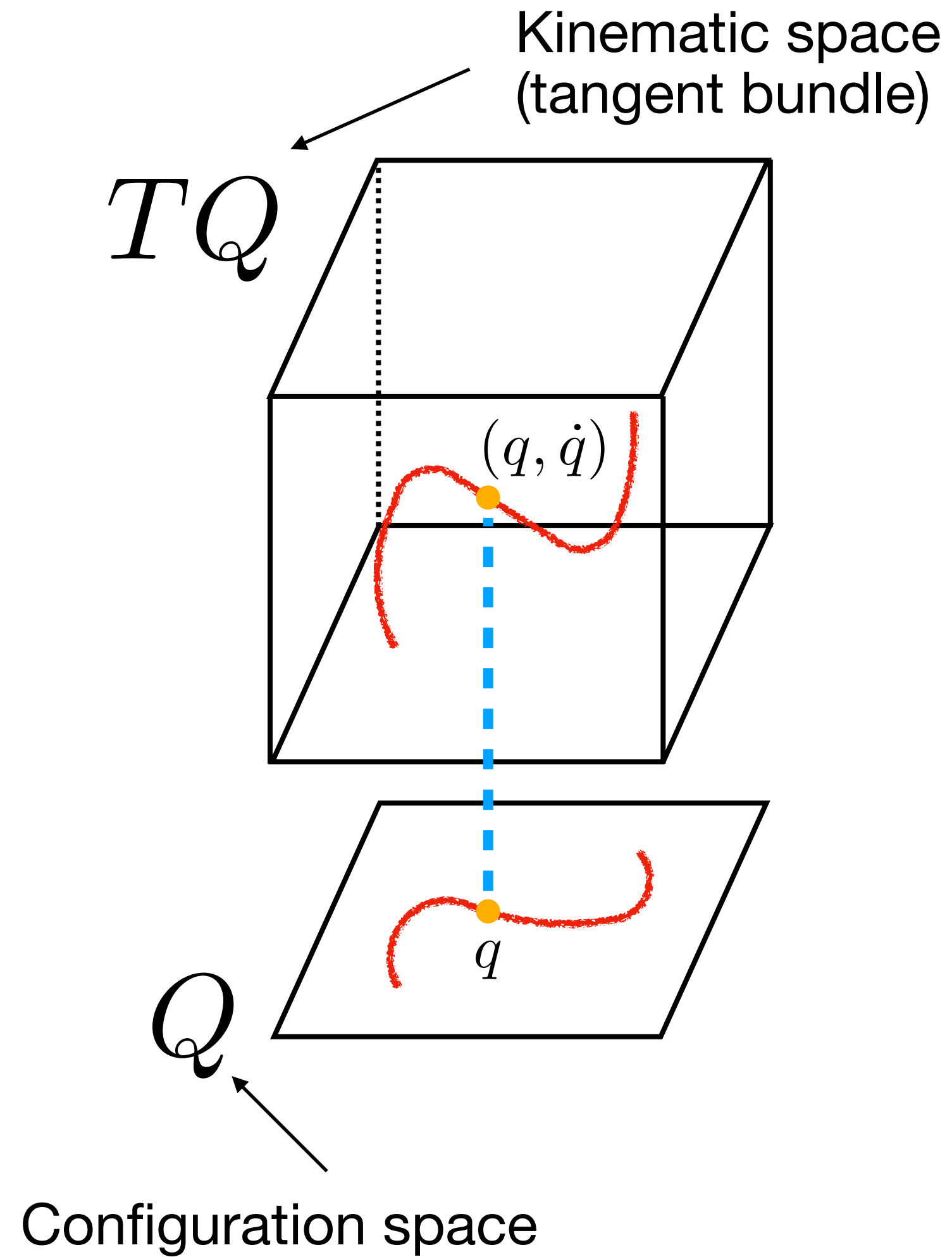
$$\arg \min_{q_t} \int_0^T \mathcal{L}(q_t, \dot{q}_t) dt \quad \longrightarrow \quad \vec{F} = m\vec{a}$$

Position $\nearrow$

Velocity $\nearrow$

Cost (Lagrangian) $\nearrow$

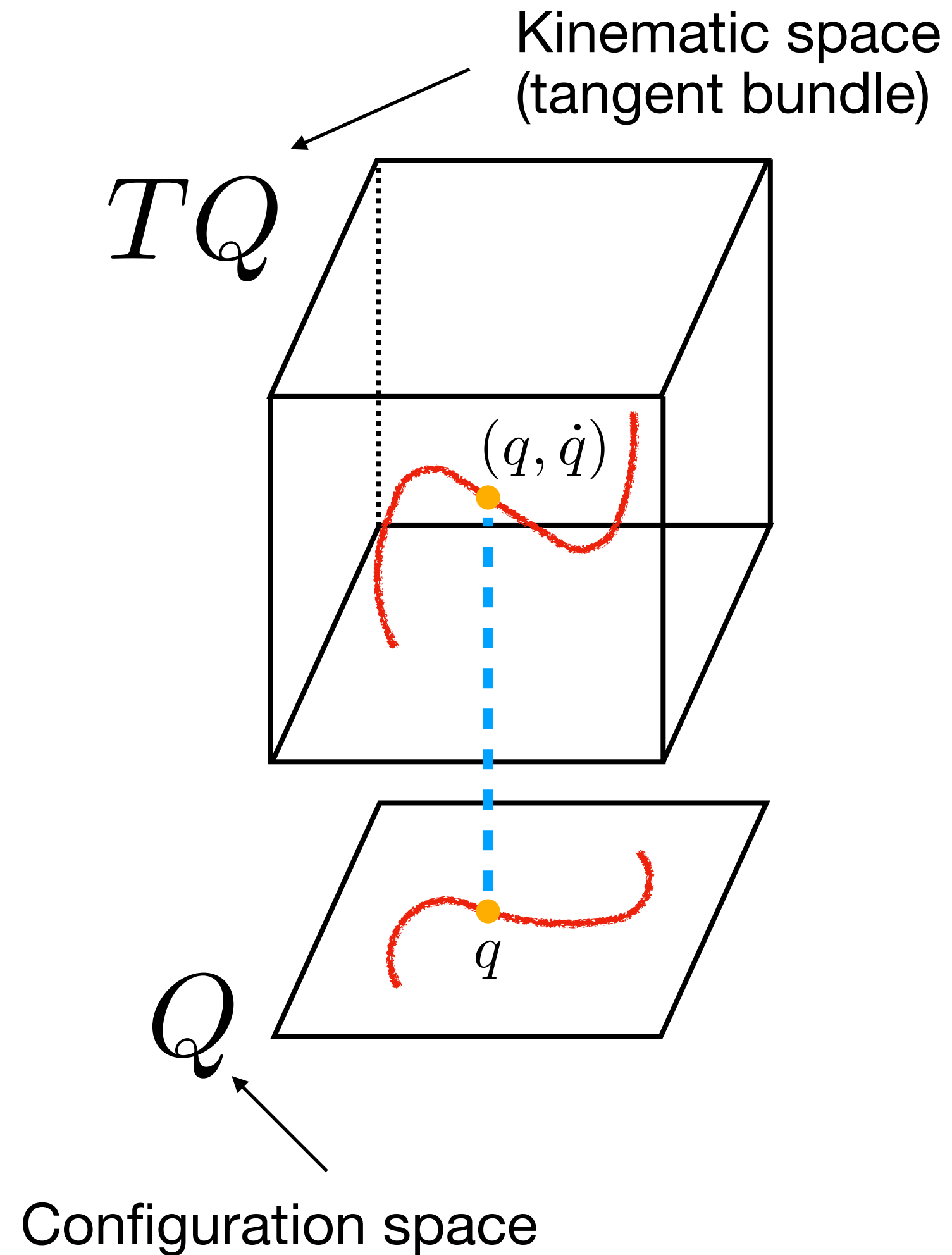
Lagrangian Mechanics



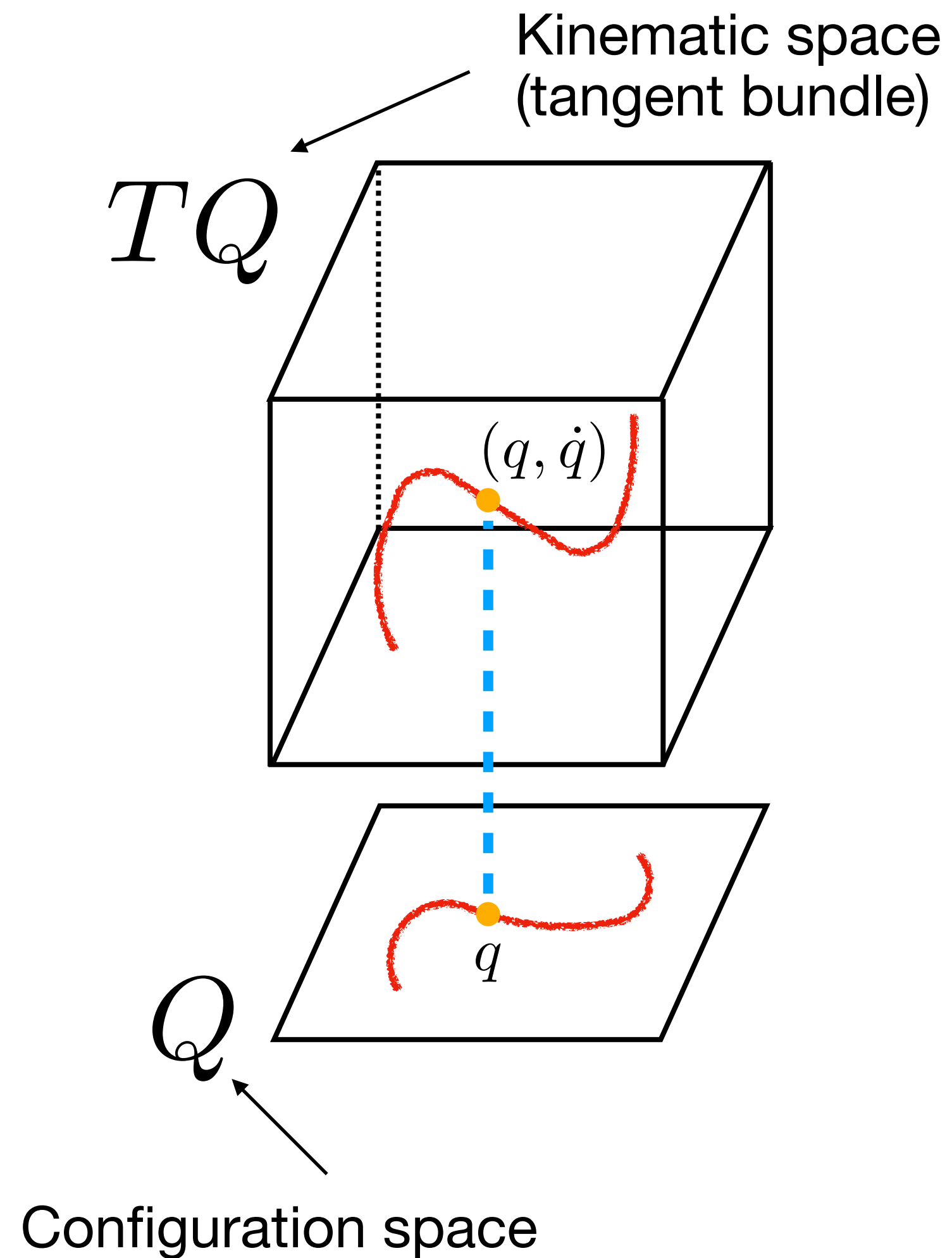
Lagrangian Mechanics

Position Velocity

$$\mathcal{L}(q_t, \dot{q}_t) = K - U$$



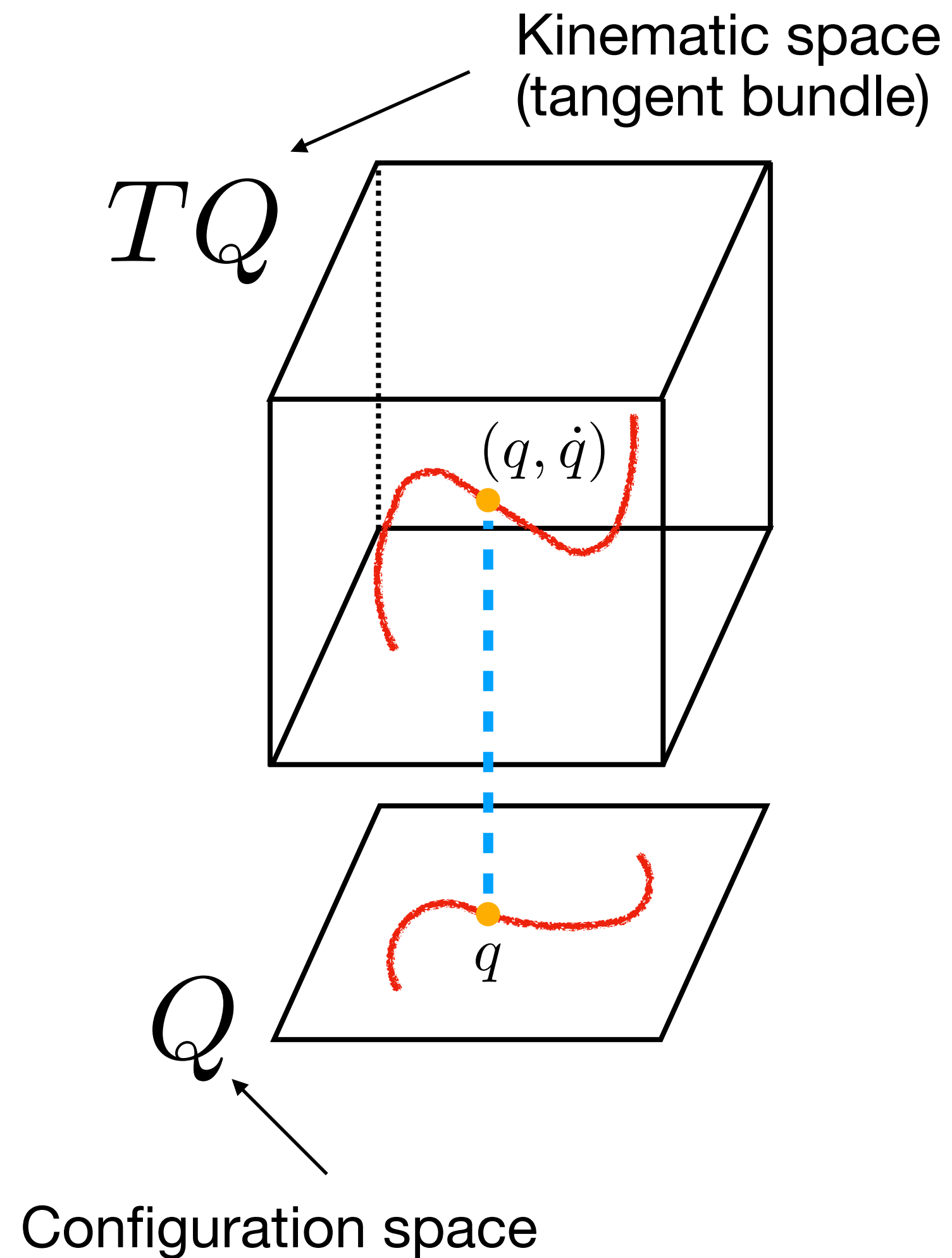
Lagrangian Mechanics



$$\mathcal{L}(q_t, \dot{q}_t) = K - U$$

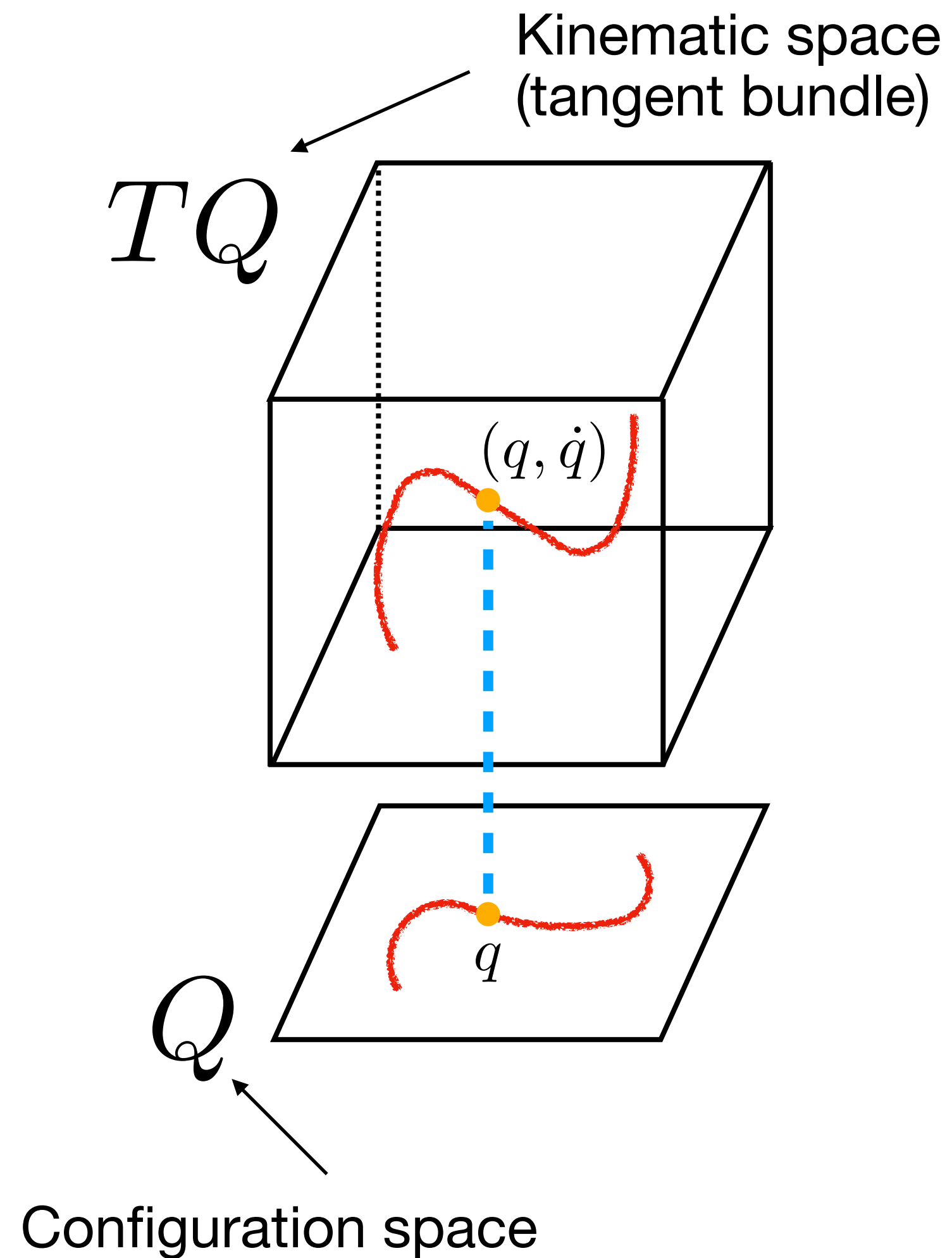
Kinetic energy $\frac{1}{2}mv^2$
Potential energy

Lagrangian Mechanics



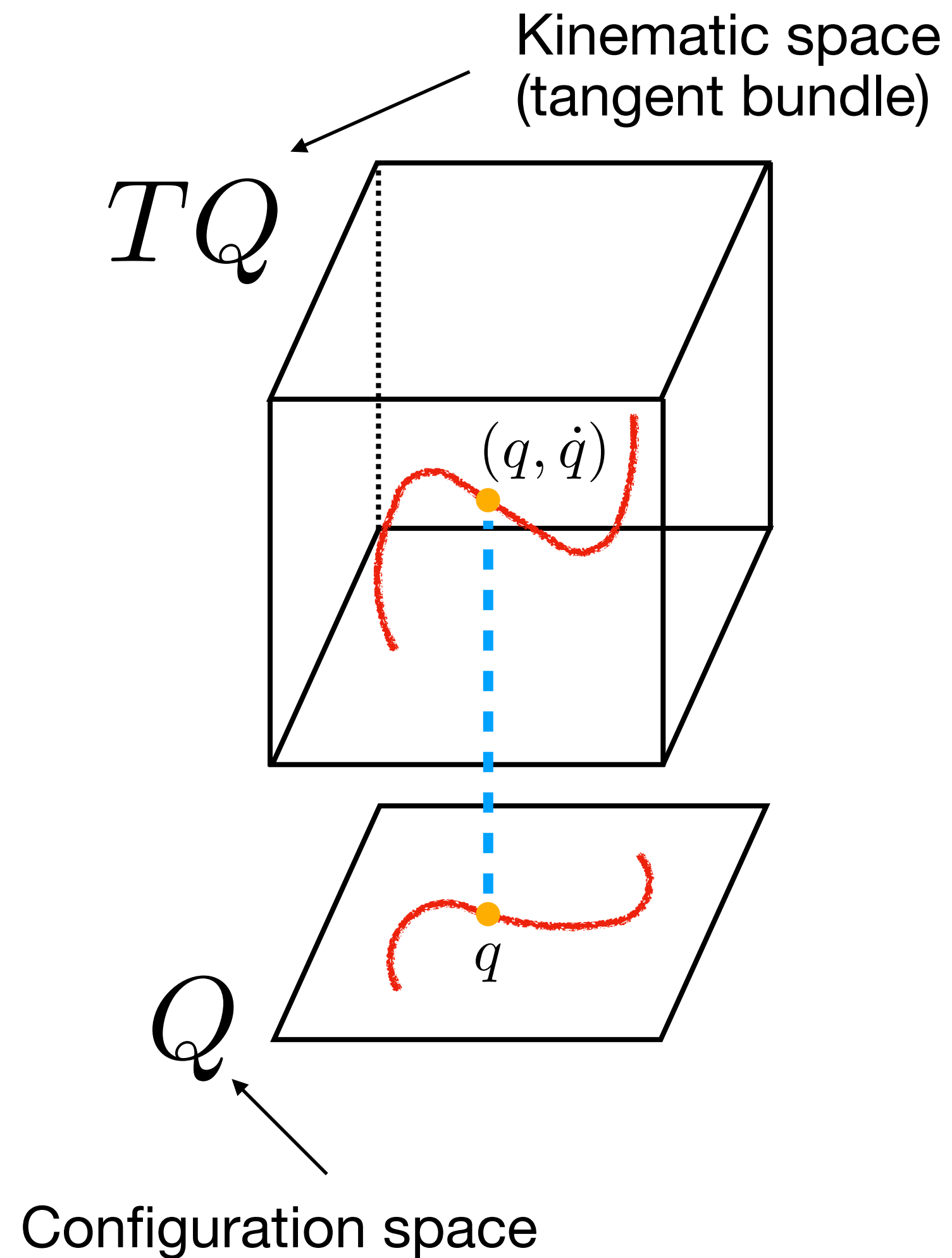
$$S(q) = \int_0^T \mathcal{L}(q_t, \dot{q}_t) dt$$

Lagrangian Mechanics



$$\arg \min_{q_t} S(q) = \int_0^T \mathcal{L}(q_t, \dot{q}_t) dt$$

Lagrangian Mechanics

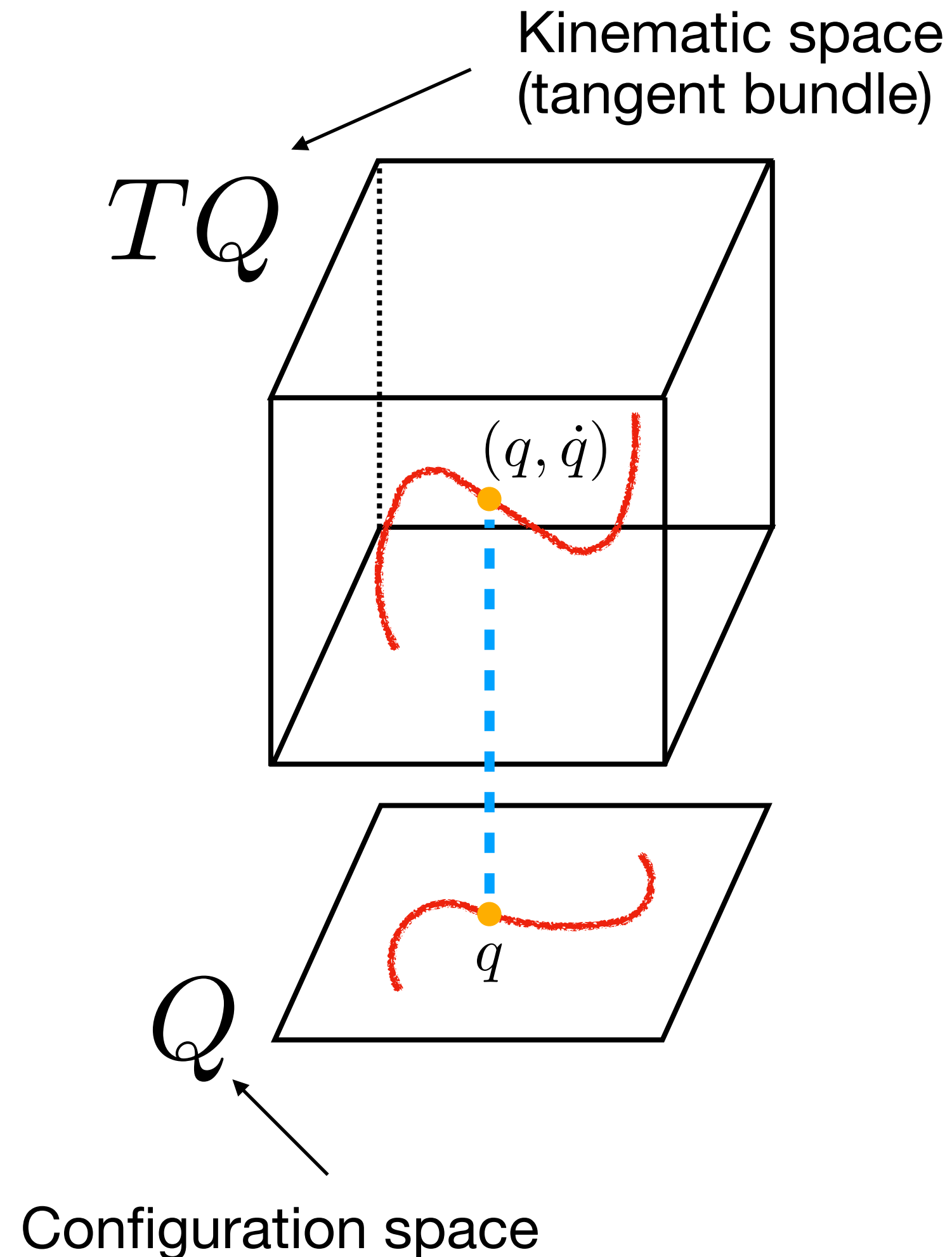


$$\arg \min_{q_t} S(q) = \int_0^T \mathcal{L}(q_t, \dot{q}_t) dt$$

$$\frac{\delta S(q)}{\delta q} = 0$$

Hamilton's Principle of Stationary Action

Lagrangian Mechanics



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Hamilton's Principle of Stationary Action

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}$$

Euler-Lagrange Equations

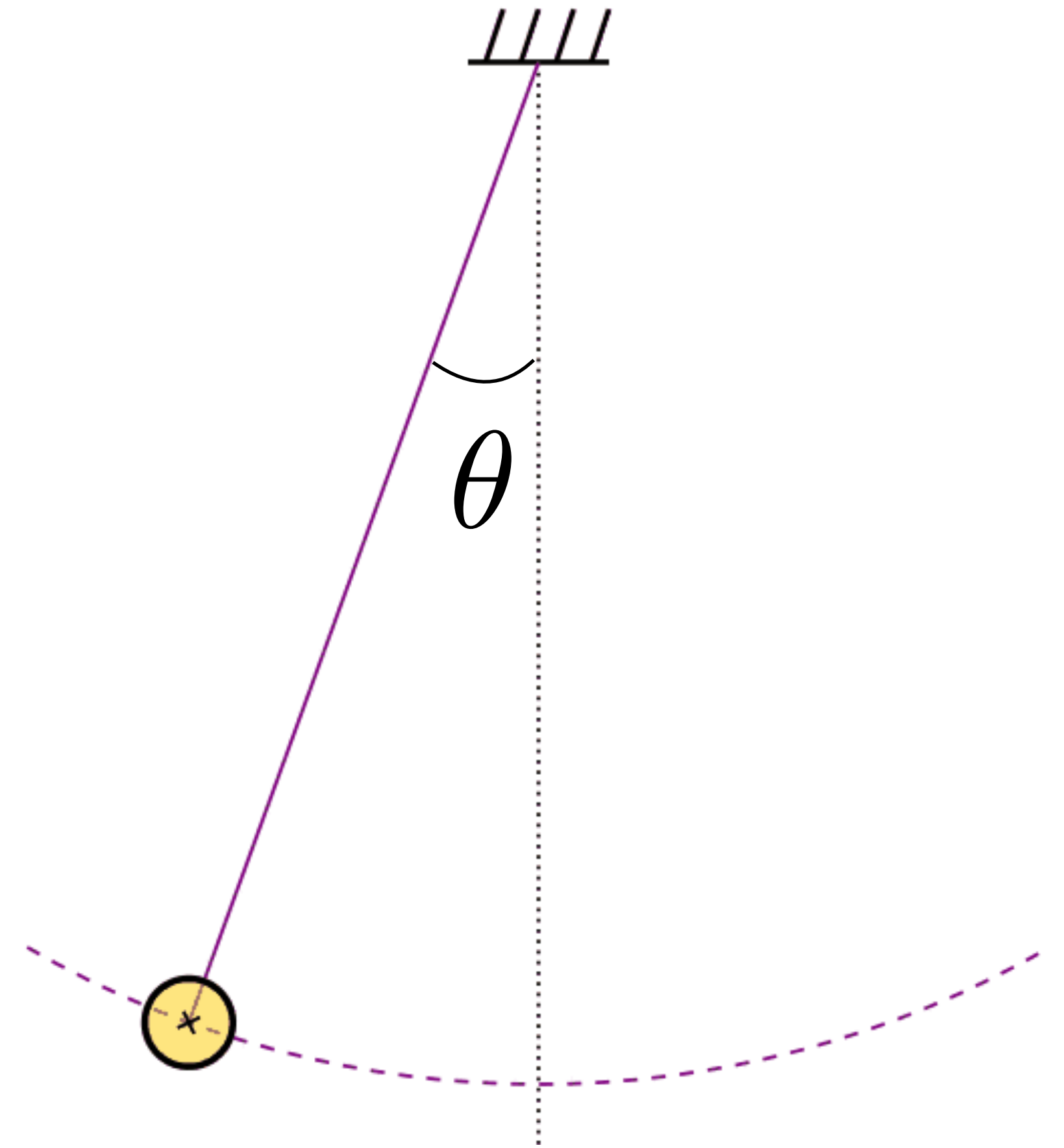
Lagrangian Mechanics

Example: pendulum motion

Configuration space: circle $Q = S^1$

Position: $(\ell \sin \theta, -\ell \cos \theta)$

Speed: $\ell \dot{\theta}$



[Stündle, Animation einer Pendelschwingung, 2011]

Lagrangian Mechanics

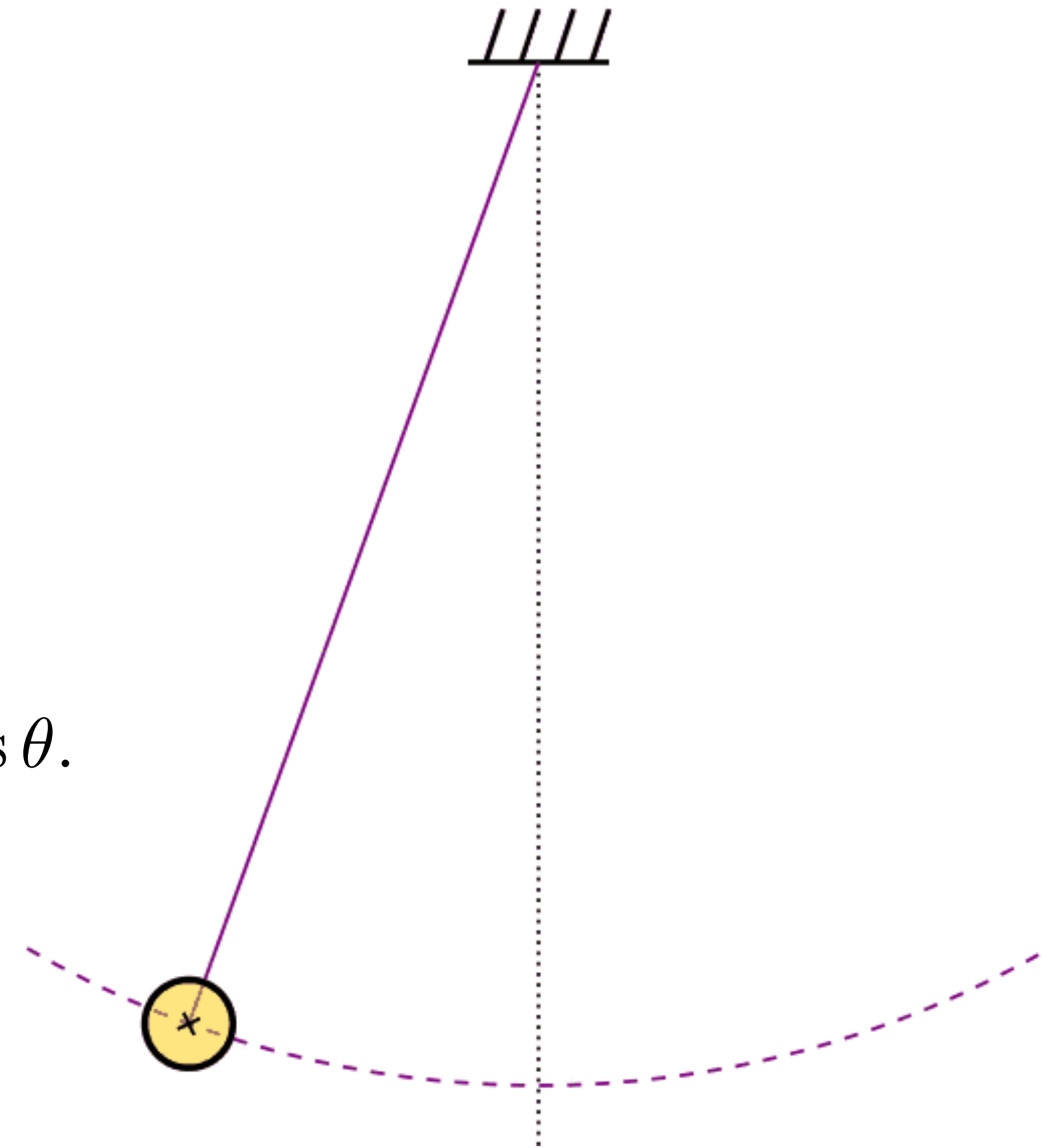
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$$K = \frac{1}{2} m \ell^2 \dot{\theta}^2, \quad U = -m g \ell \cos \theta, \quad L = \frac{1}{2} m \ell^2 \dot{\theta}^2 + m g \ell \cos \theta.$$



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Lagrangian Mechanics

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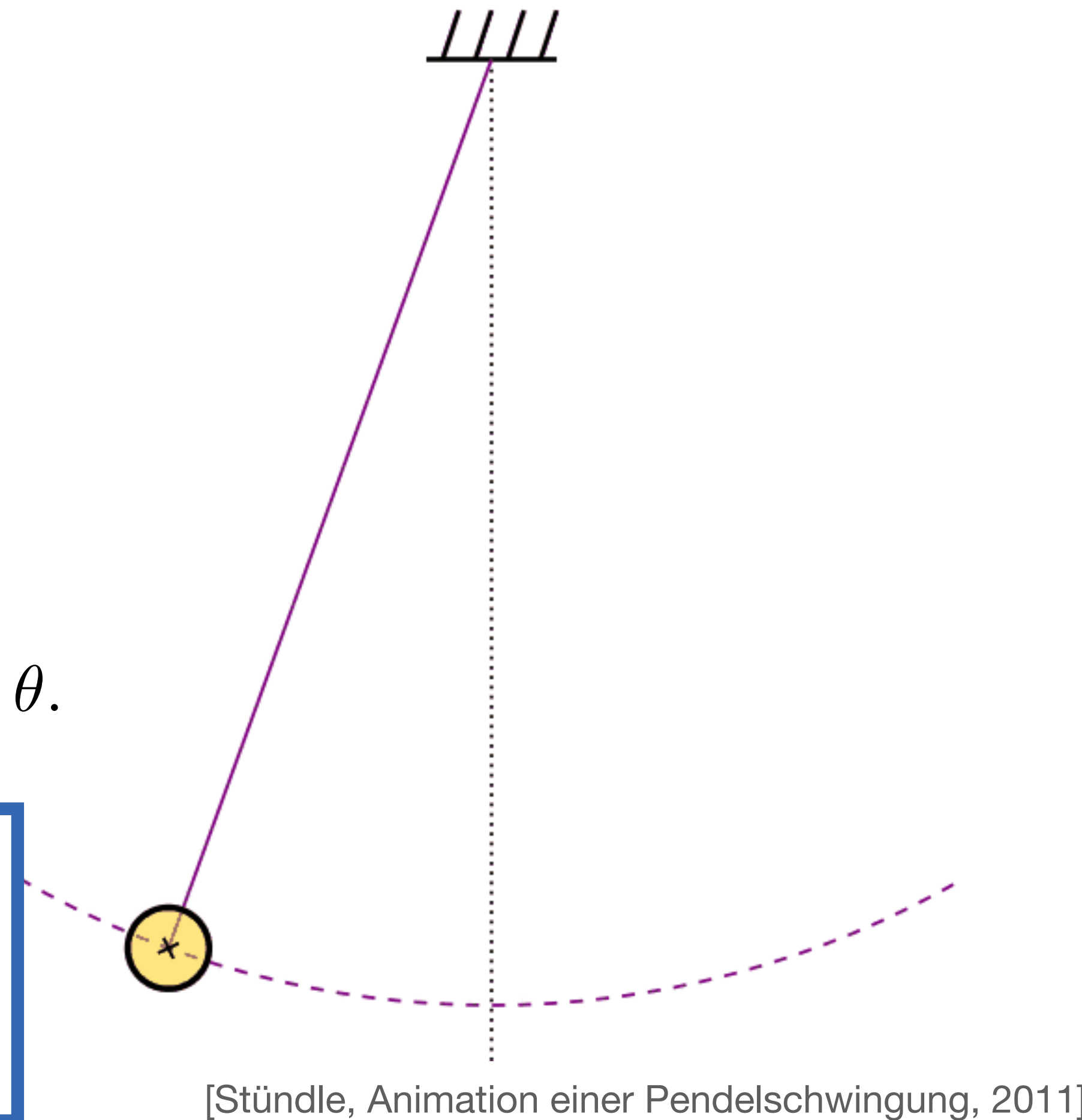
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Lagrangian Mechanics

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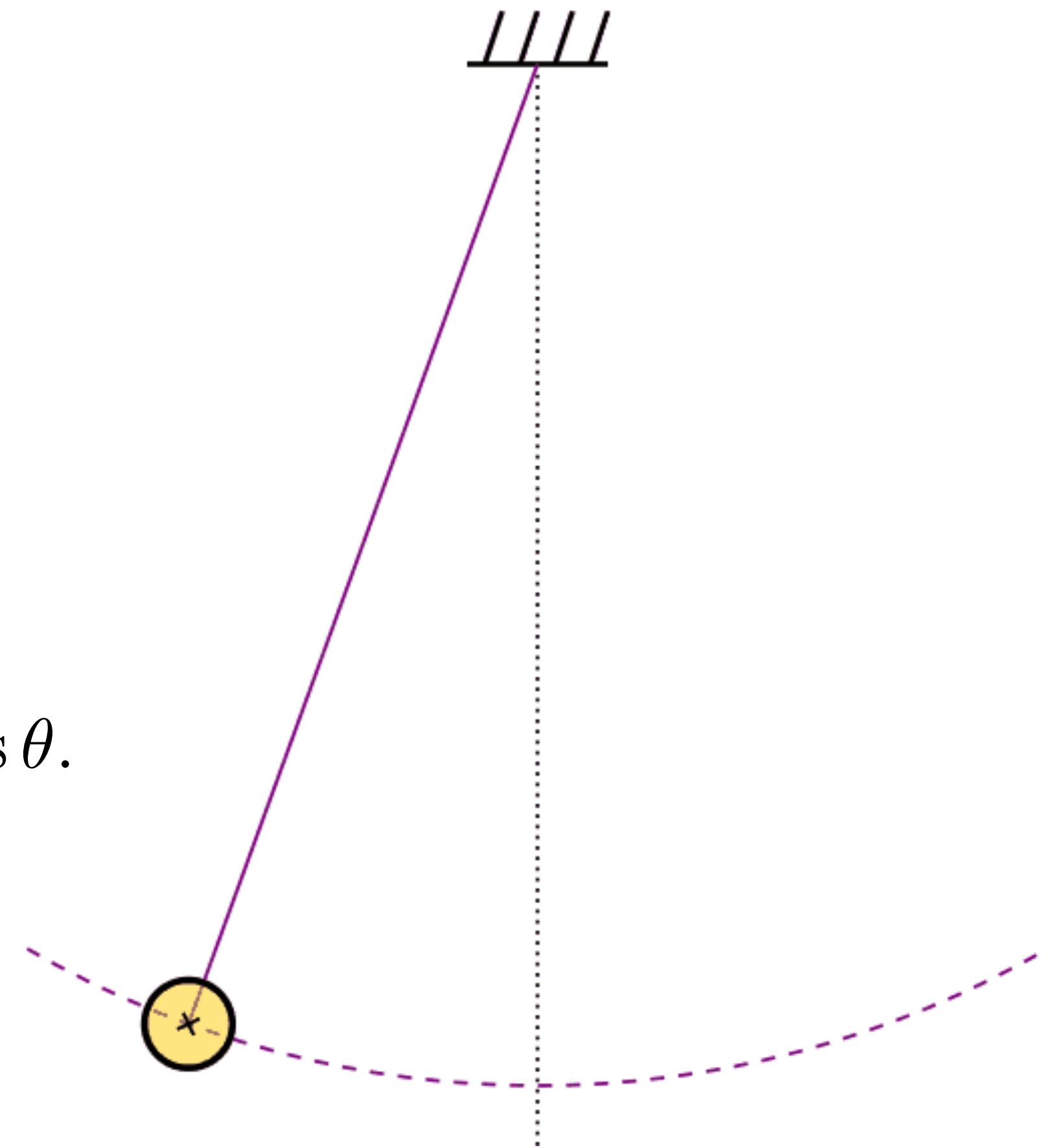
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Equation of motion

$$\frac{d}{dt}(m \ell^2 \dot{\theta}) + m g \ell \sin \theta = 0 \quad \Longleftrightarrow \quad \ddot{\theta} = -\frac{g}{\ell} \sin \theta.$$

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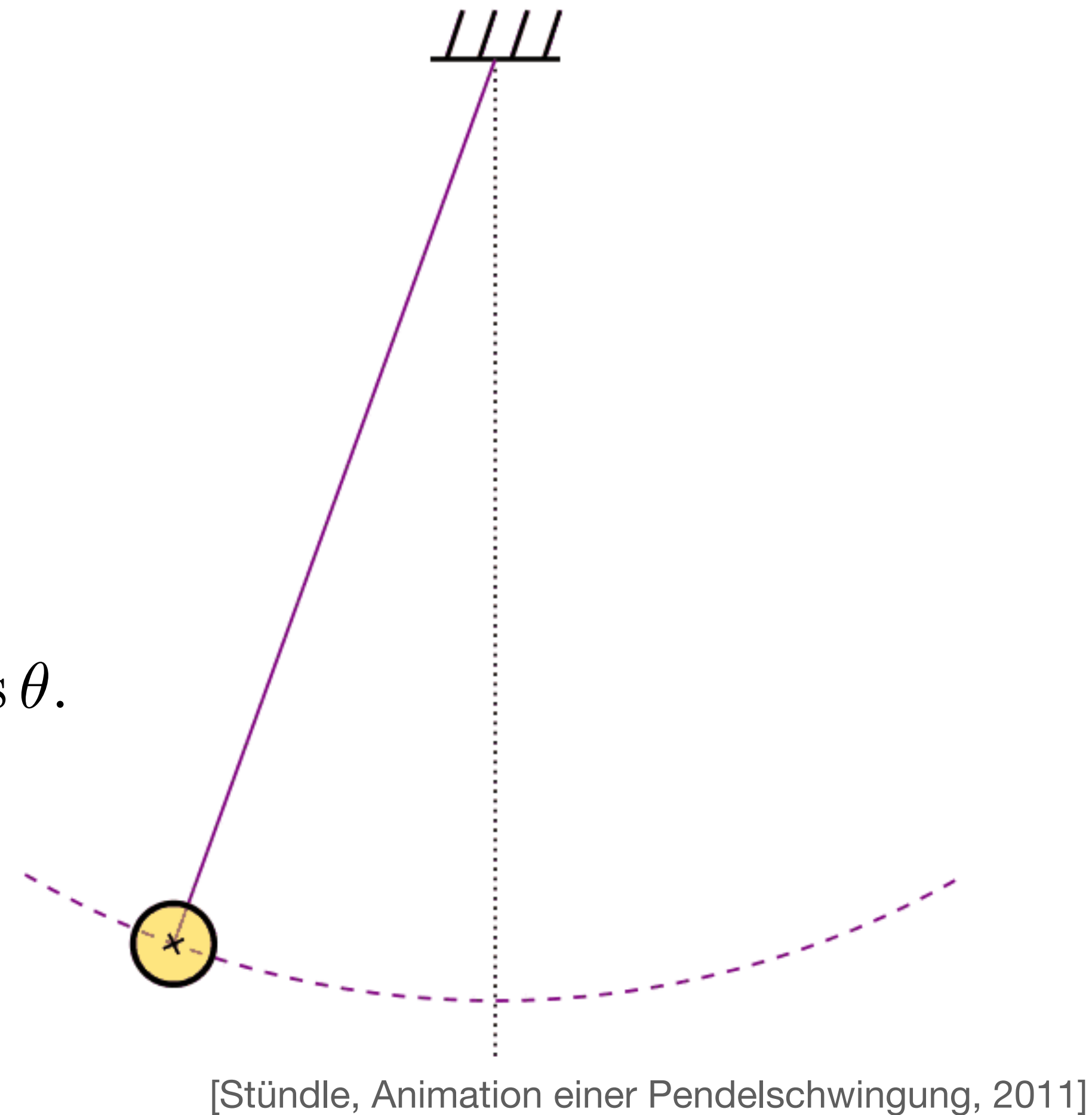
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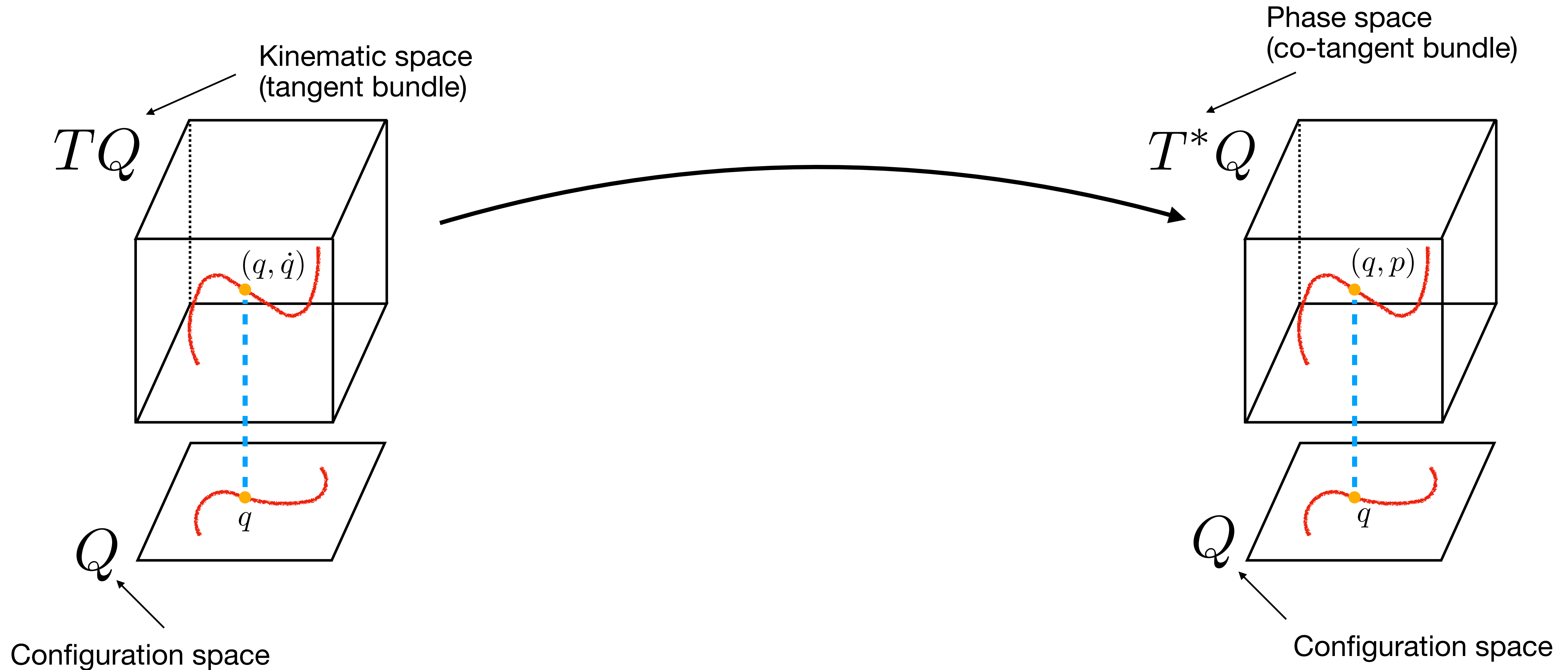
Equation of motion

Conjugate momentum

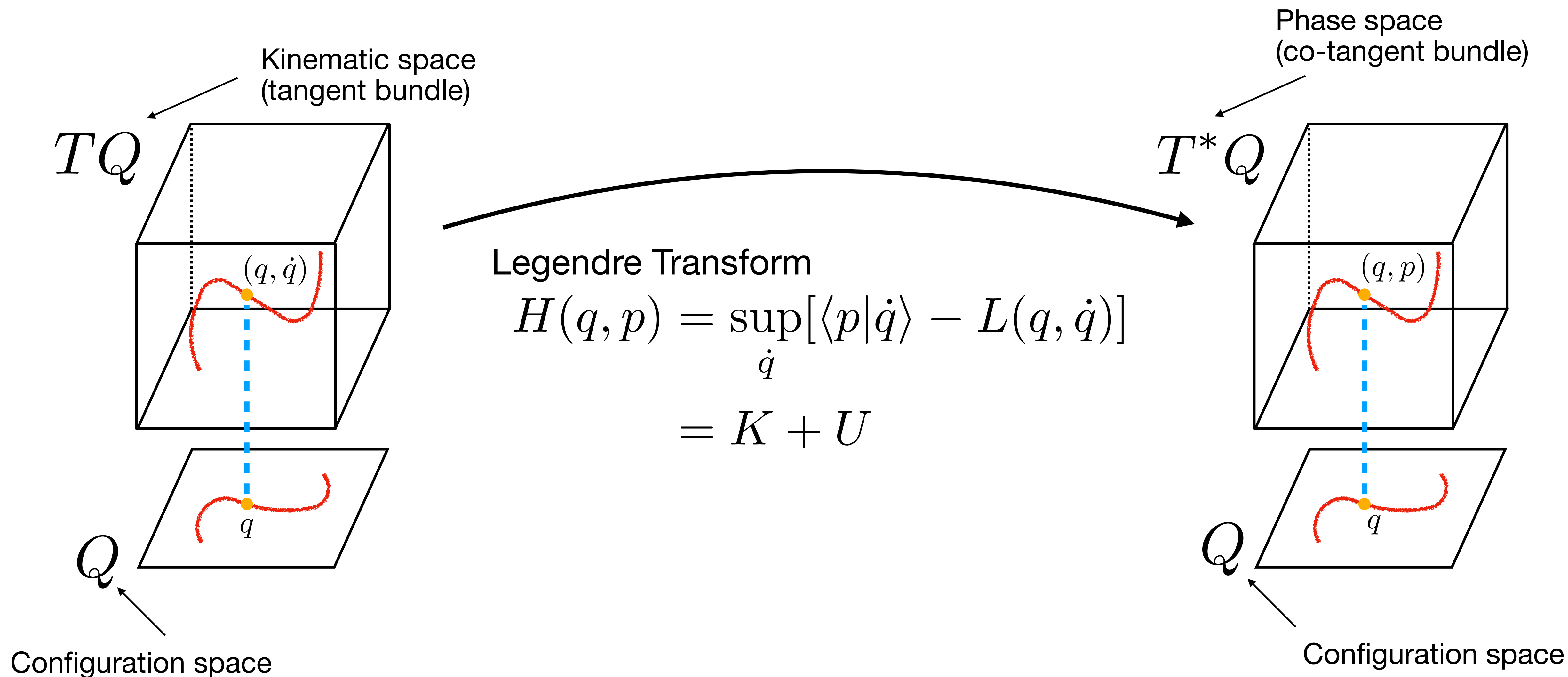
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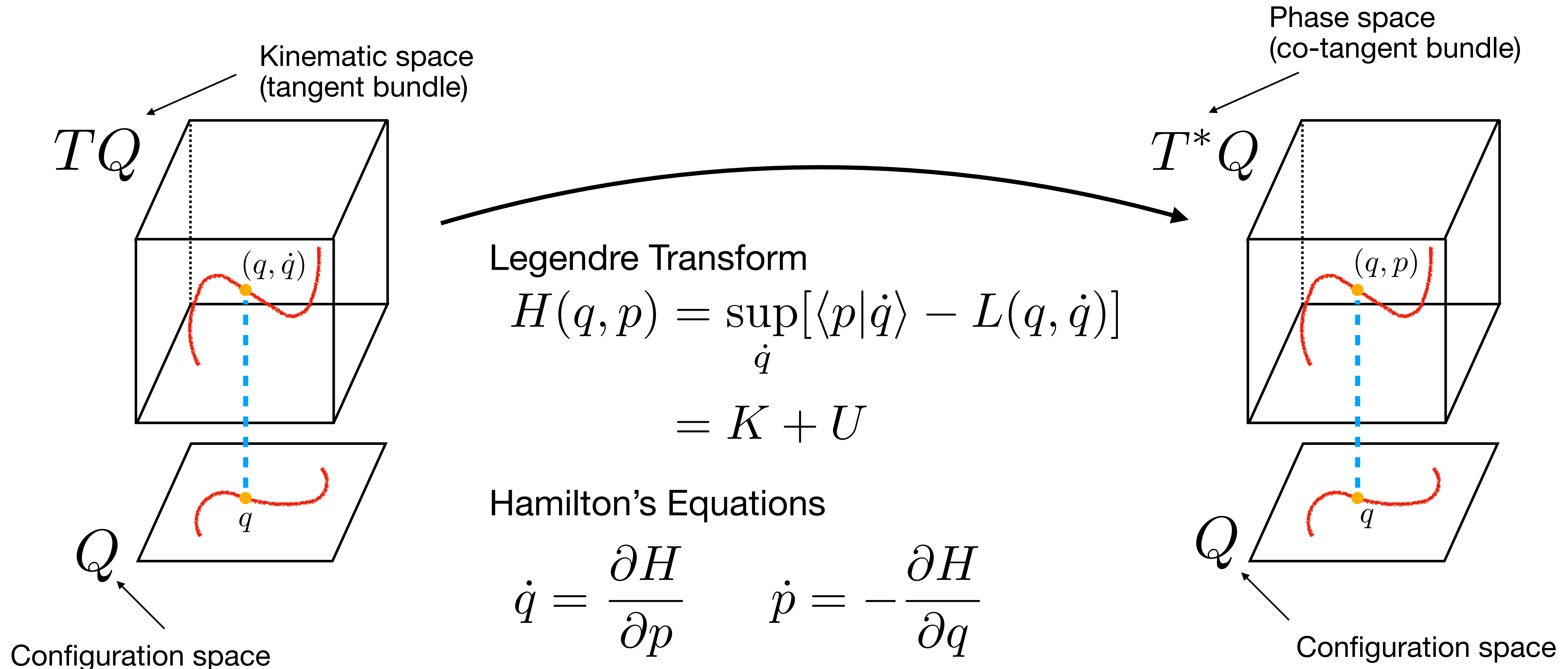
From Lagrangian to Hamiltonian Mechanics



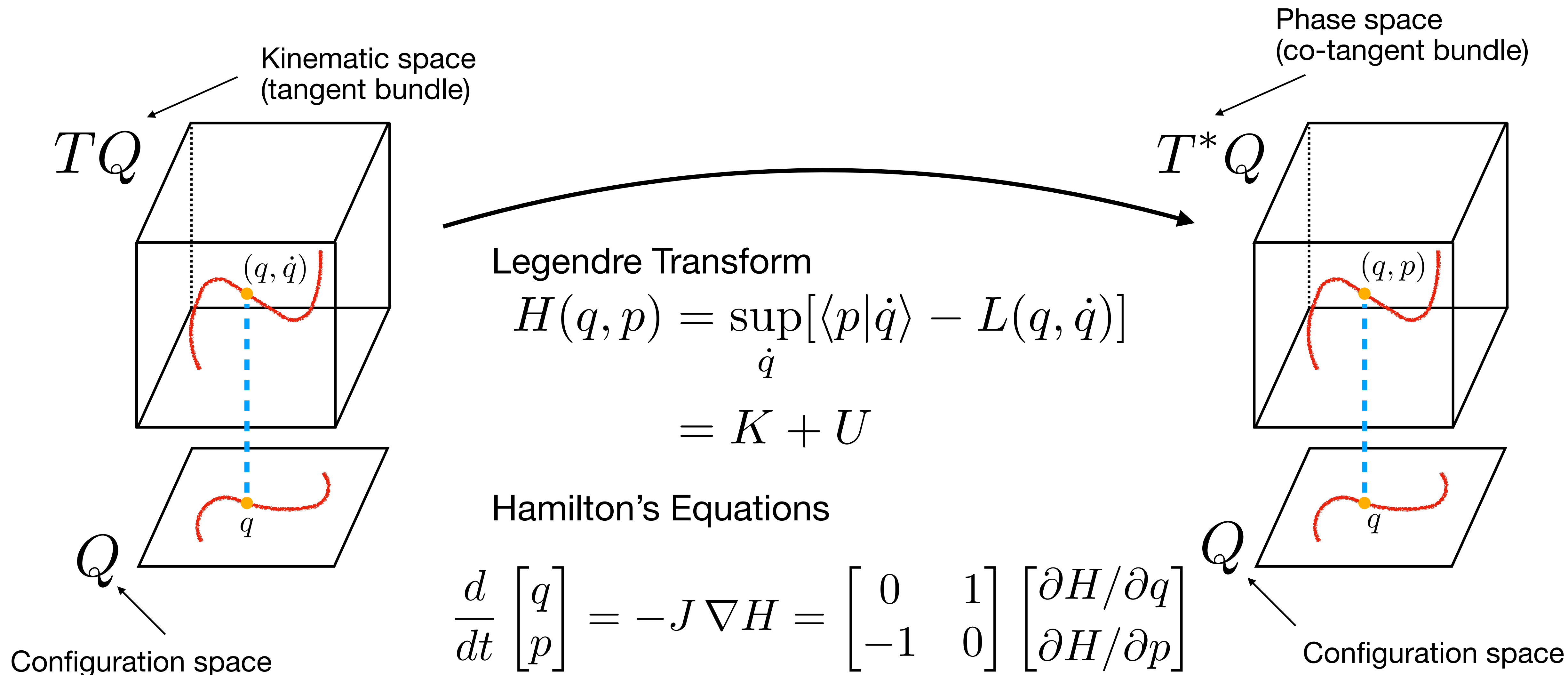
From Lagrangian to Hamiltonian Mechanics



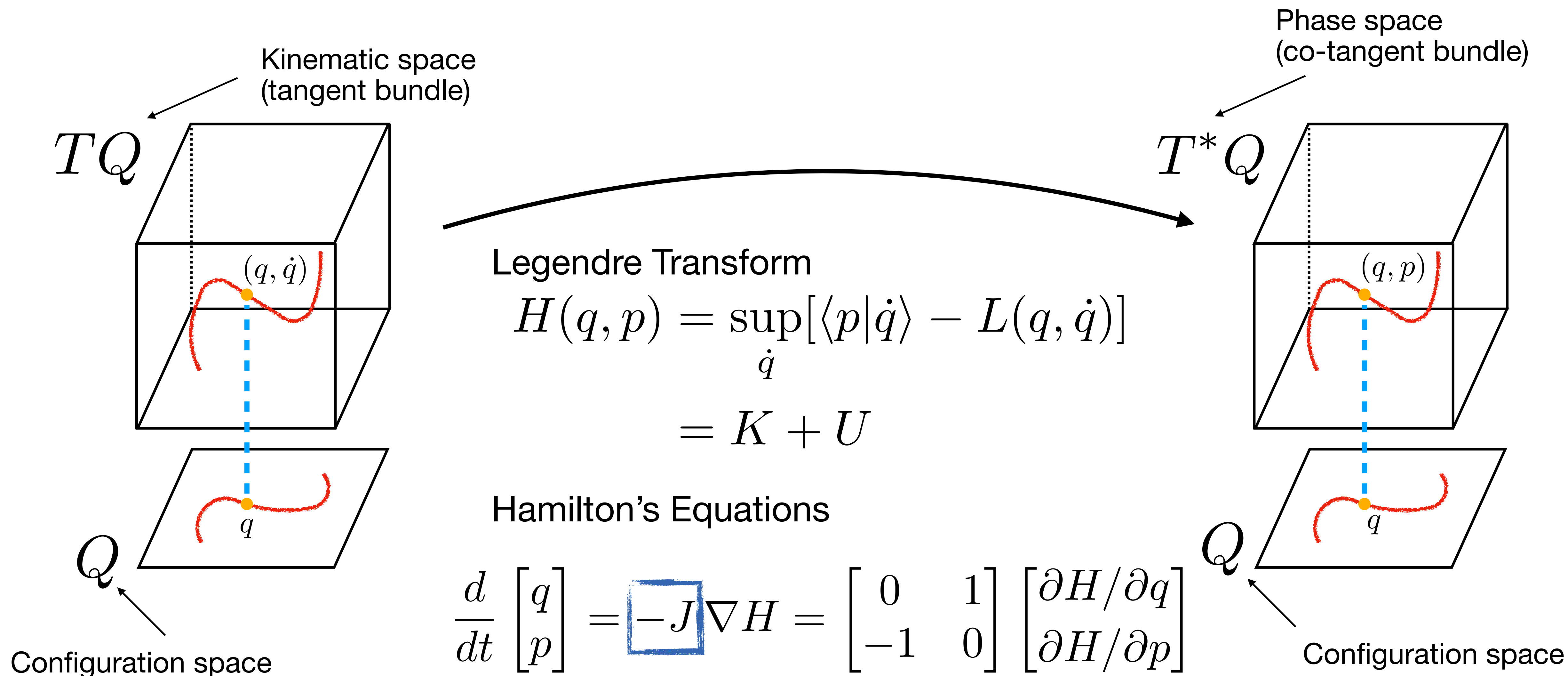
From Lagrangian to Hamiltonian Mechanics



From Lagrangian to Hamiltonian Mechanics



From Lagrangian to Hamiltonian Mechanics



Symplectic Geometry

Definition: Symplectic Manifold

A symplectic manifold is a pair (Σ, σ)

σ : a 2-form that is

(1) non-degenerate: $X \rightarrow \sigma(X, \cdot)$ is invertible

(2) closed: $d\sigma = 0$

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Symplectic Geometry

σ gives us a pair of musical isomorphisms.

$$\flat : T_x \Sigma \rightarrow T_x^* \Sigma, \quad \flat(X) = \iota_X \sigma = \sigma(X, \cdot), \quad \sharp = \flat^{-1}$$

Then the symplectic gradient of a hamiltonian H is:

$$\text{sgrad } H = \sharp(dH), \text{ or equivalently } \iota_{\text{sgrad } H} \sigma = dH.$$

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Hamiltonian flow

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Hamiltonian flow

i.e., the solutions to $\dot{x} = -\text{sgrad } H$

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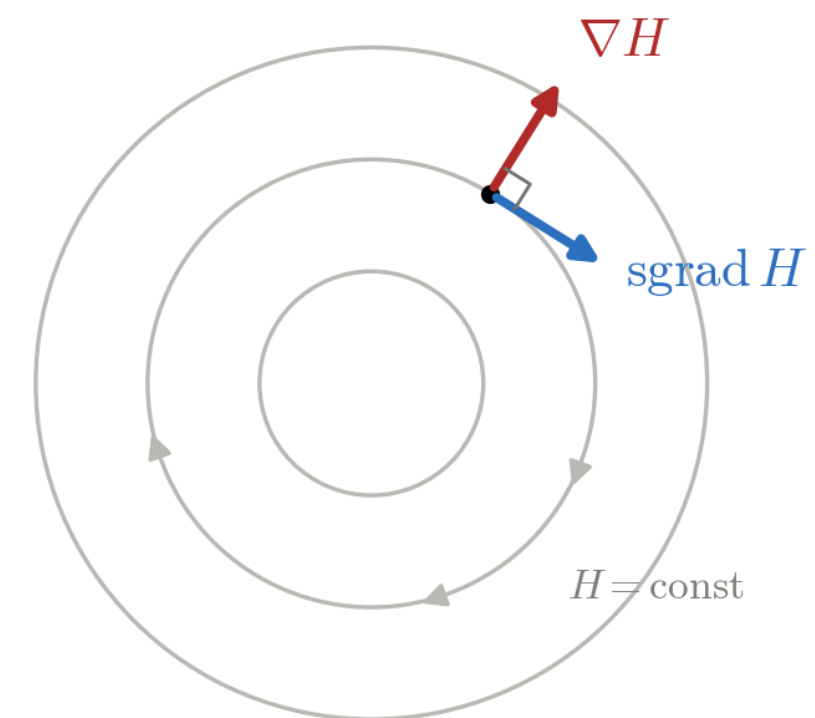
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Symplectic Geometry

Liouville's Theorem

For any H , the Hamiltonian flow preserves the symplectic form:

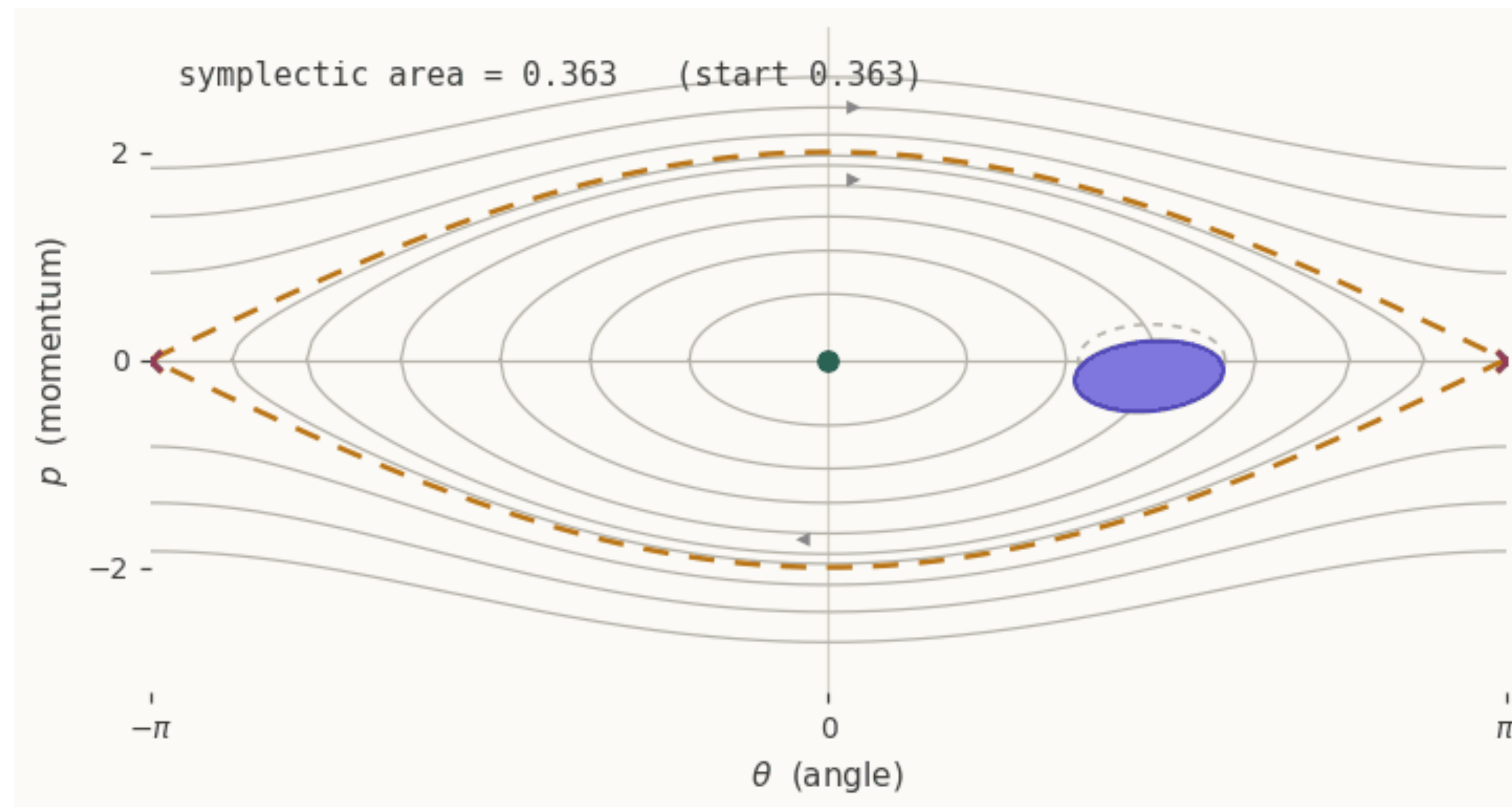
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Symplectic Geometry

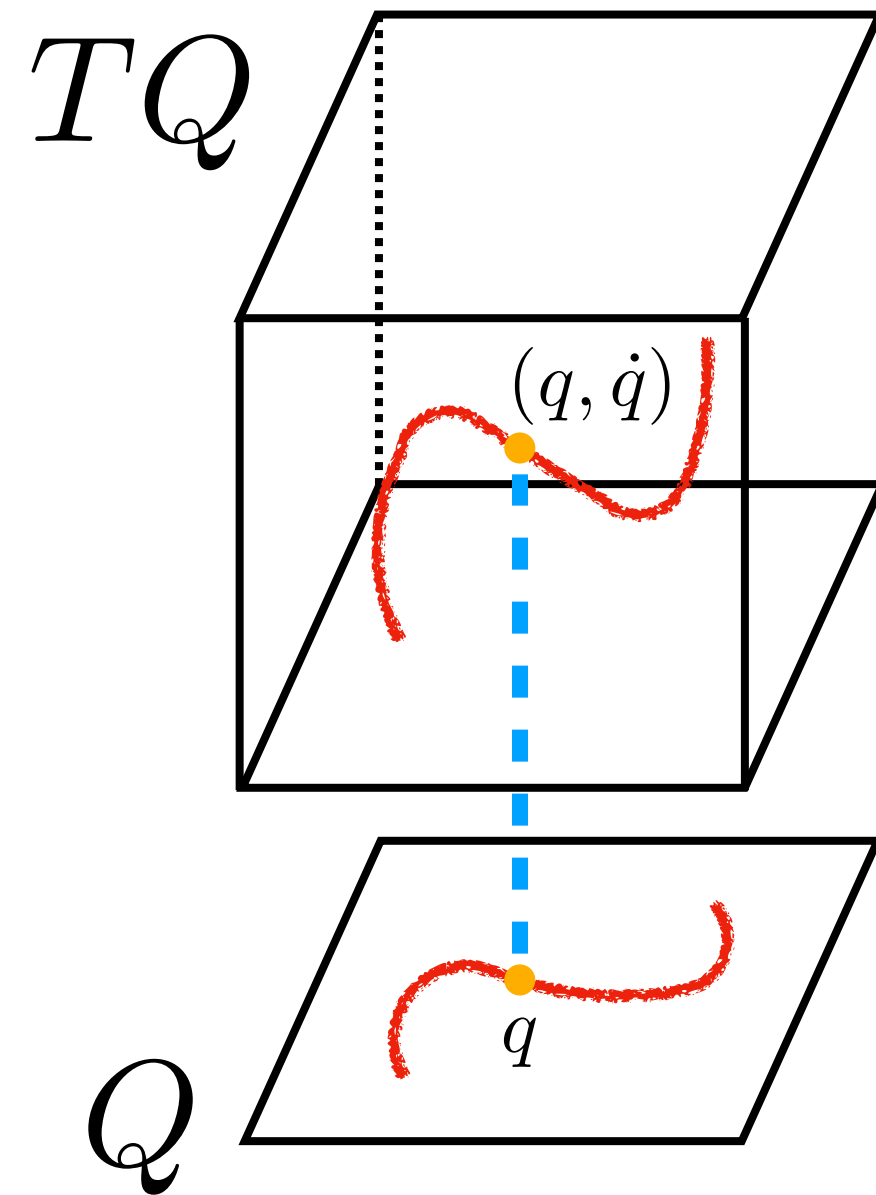
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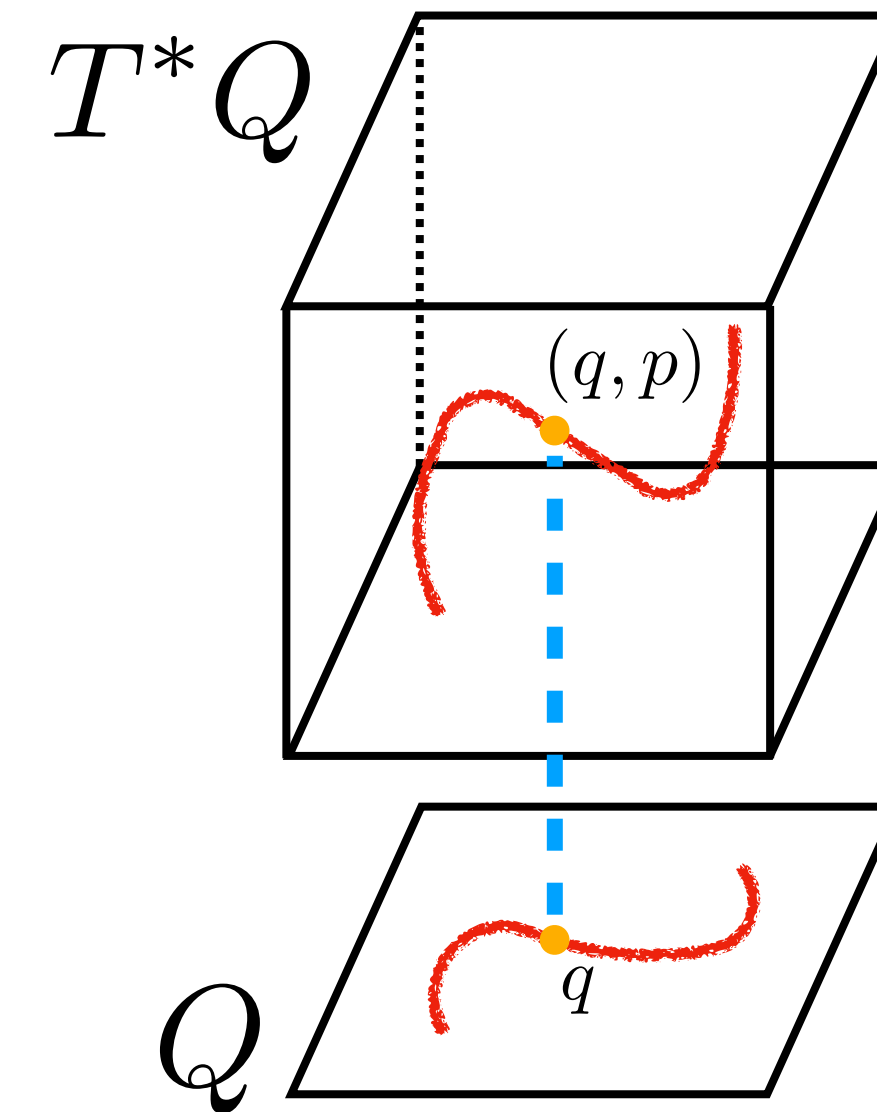


Summary



Lagrangian Mechanics

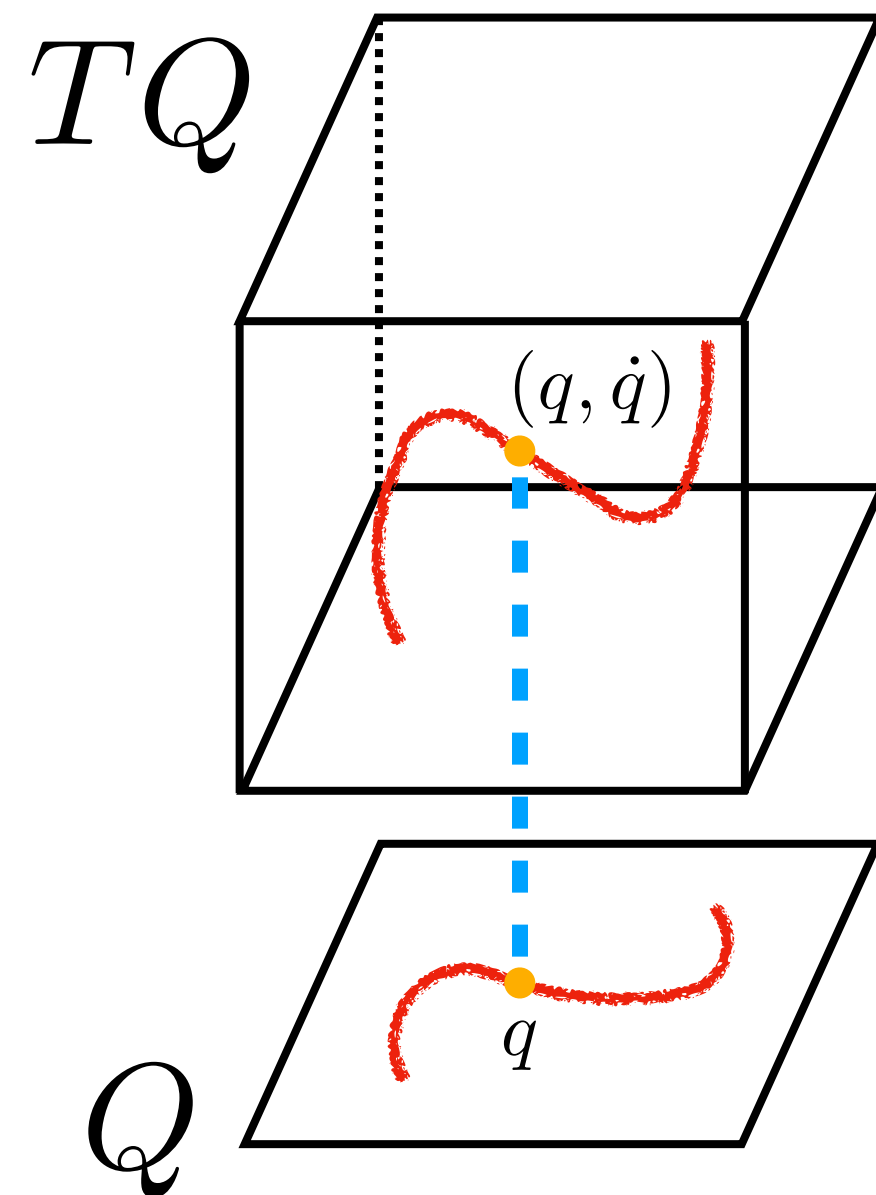
$$\mathcal{L}(q, \dot{q}) = K - U$$



Hamiltonian Mechanics

$$\mathcal{H}(q, p) = K + U$$

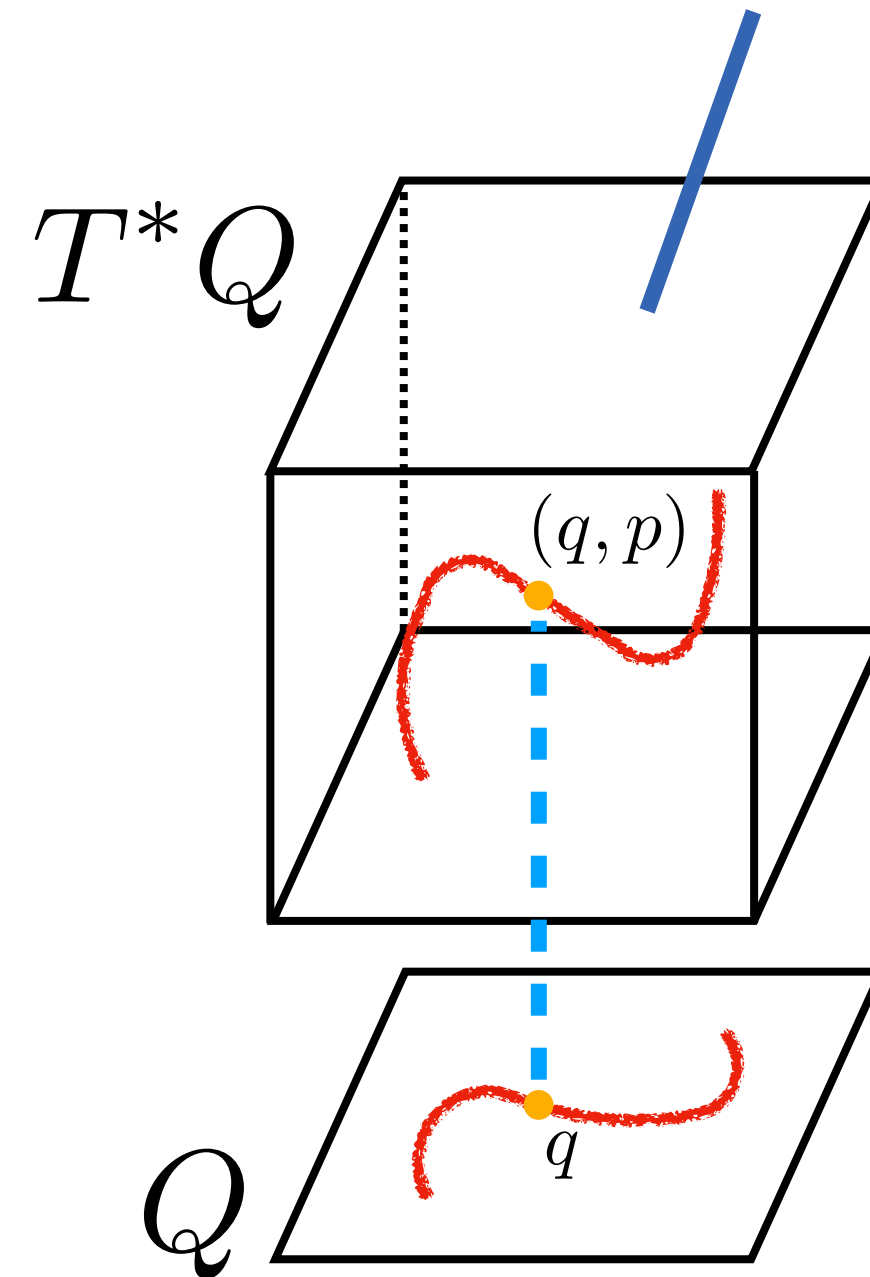
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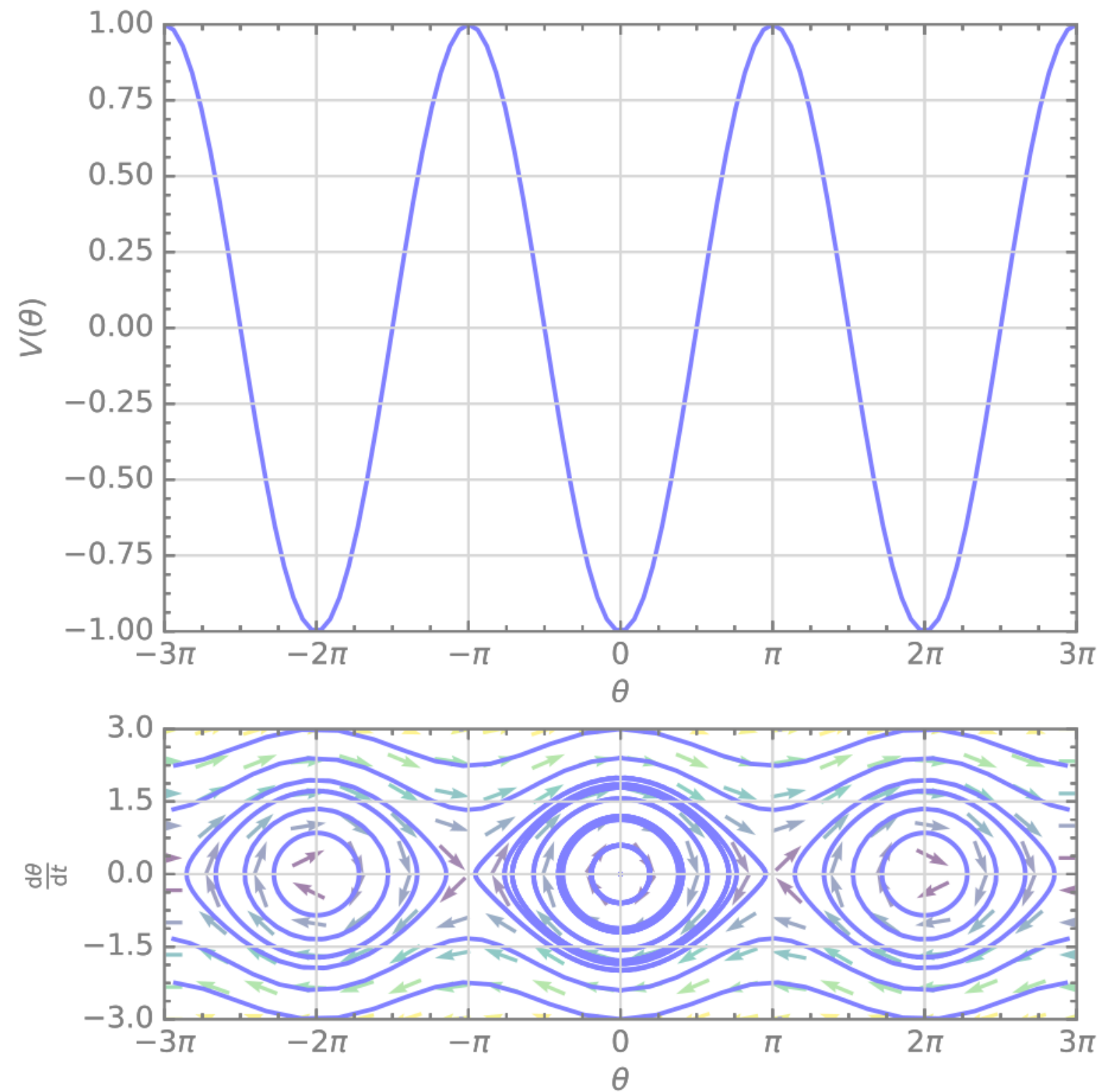
Geometric Structures



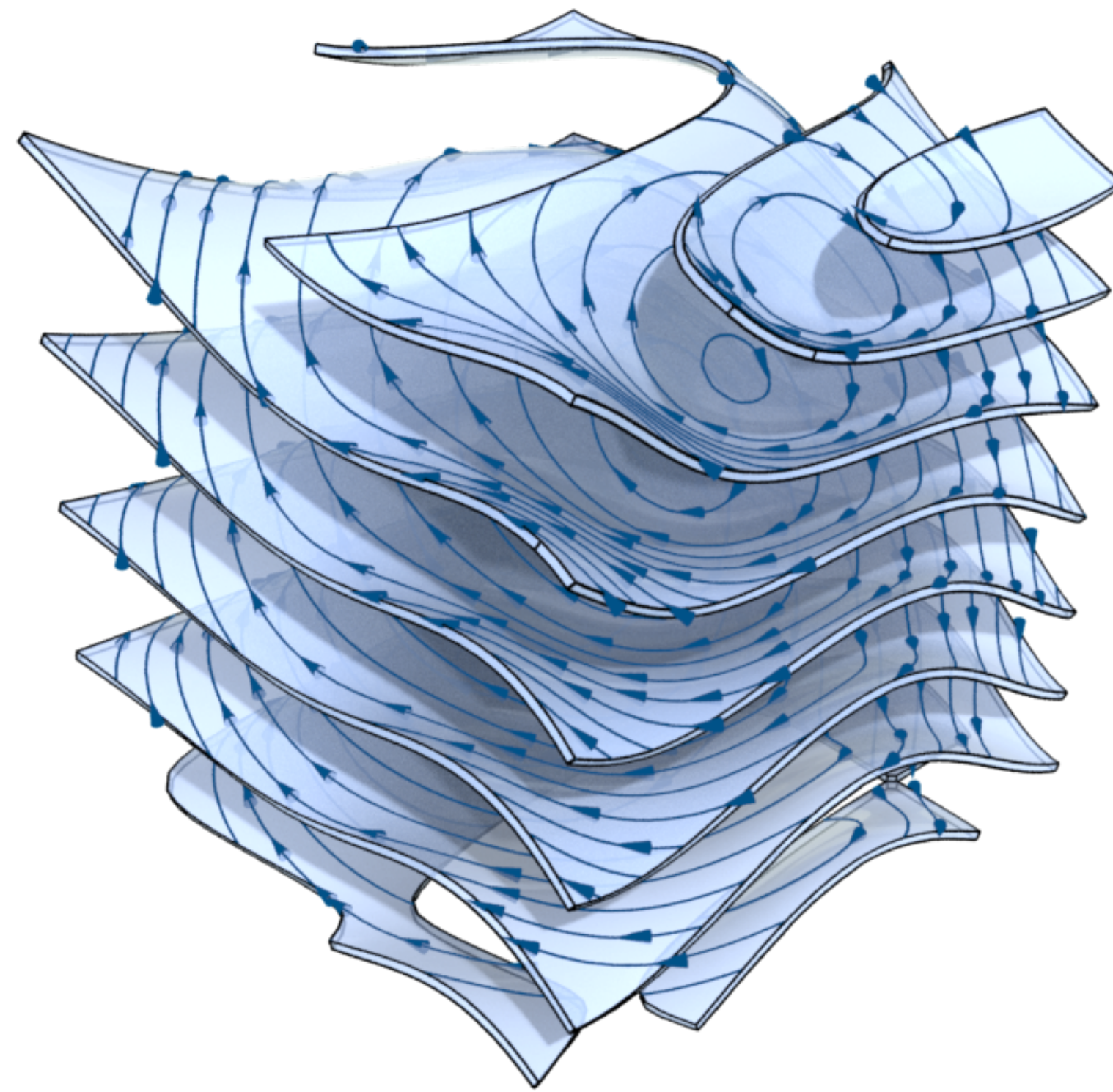
Hamiltonian Mechanics

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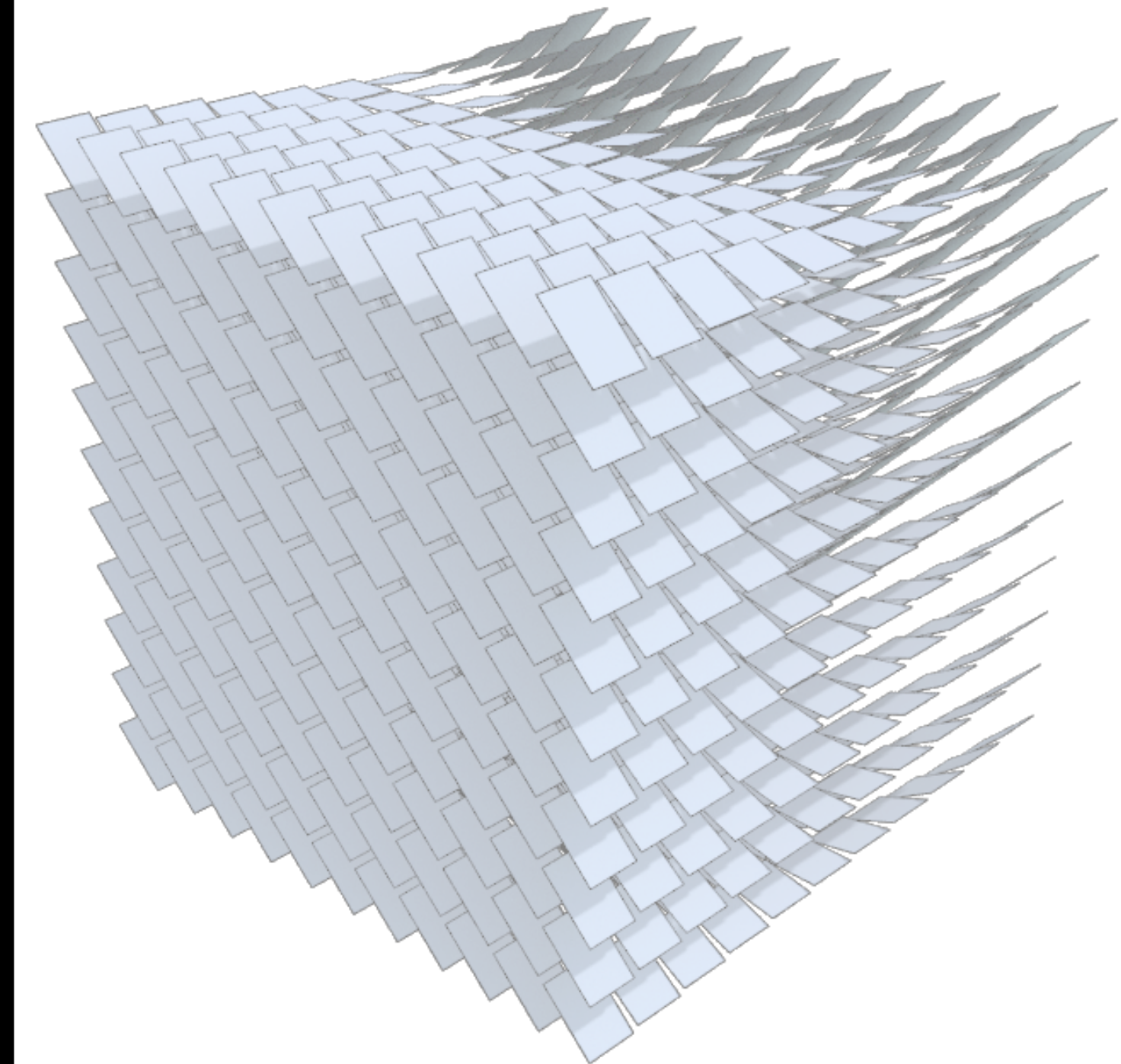
Overview



**Lagrangian and
Hamiltonian Mechanics**



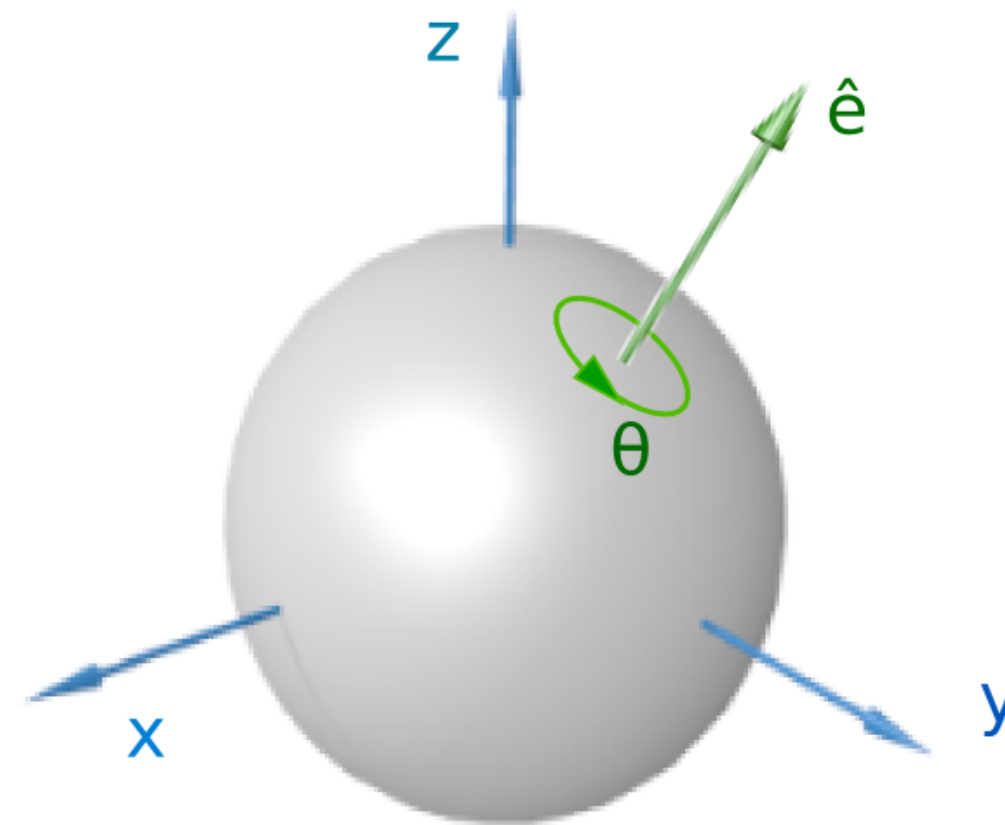
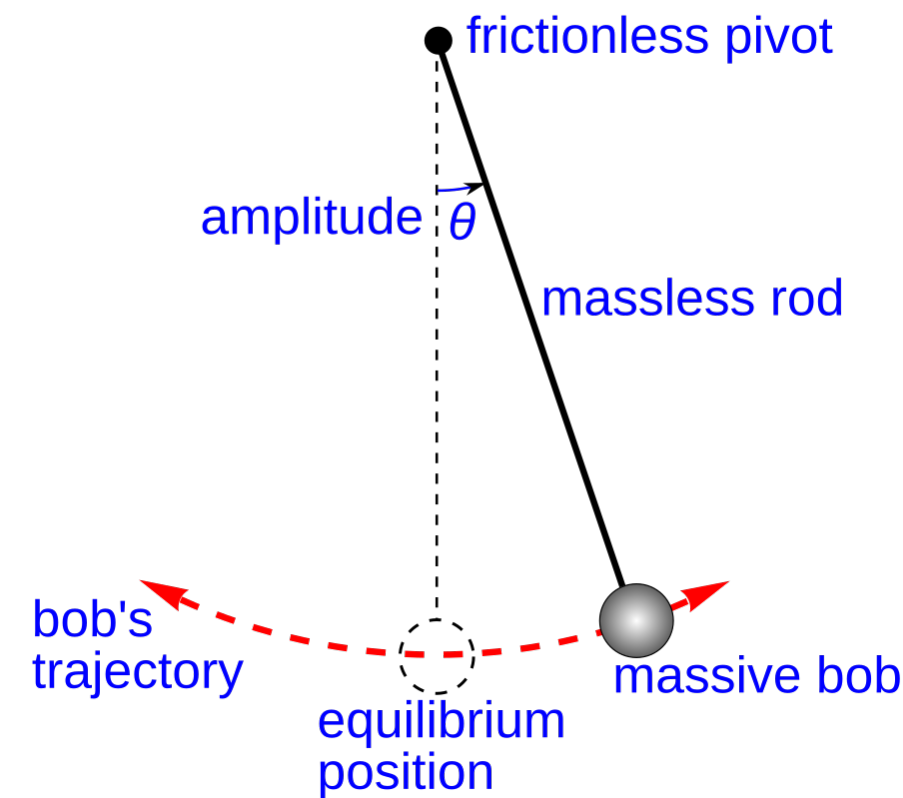
**Lie groups, Symmetry,
and Reduction**



**Discretization as a
constrained variational
problem**

Physical change of variables

Lie groups capture then



- All described using a change coordinates map $g : M \rightarrow W$
- For example: $SO(2)$, $SO(3)$, or $\text{Diff}_{\text{vol}}(\mathbb{R}^n)$

Pendulum

$$\frac{d}{dt} (m\ell^2 \dot{\theta}) = 0$$

Euler's equation for rigid bodies

$$\frac{d}{dt} (\mathbb{I}\Omega) = \mathbb{I}\Omega \times \Omega$$

Euler's equation for ideal fluids

$$\frac{\partial}{\partial t} \eta = -\mathcal{L}_u \eta$$

$$u = \mathbb{P}_{\text{div}=0}(\eta^\#)$$

Conjugate momentum

Lie group

Definitions

Definition (Lie group)

A **Lie group** is a group G that is also a smooth manifold, such that the multiplication map $G \times G \rightarrow G$, $(g, h) \mapsto gh$, and the inversion map $G \rightarrow G$, sending $g \mapsto g^{-1}$, are smooth.

- Example: $\mathrm{SO}(3) = \{R \in \mathbb{R}^{3 \times 3} : R^\top R = I, \det R = 1\}$
- Other examples: $\mathrm{SO}(2) \cong S^1$ or $\mathrm{GL}(n, \mathbb{R})$ or $(\mathbb{R}^n, +)$

Lie algebra

Definitions

Definition (Lie algebra)

A **Lie algebra** is a vector space $\mathfrak{g}$ with a bilinear bracket $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ that is antisymmetric, $[\xi, \eta] = -[\eta, \xi]$, and satisfies the Jacobi identity

$$[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0.$$

For a matrix group this bracket is the commutator $[A, B] = AB - BA$. The **Lie algebra of a Lie group** G is its tangent space at the identity, $\mathfrak{g} = T_e G$.

- Example: $\mathfrak{so}(3) = \{\Omega \in \mathbb{R}^{3 \times 3} : \Omega^\top = -\Omega\}$

Lie group action

Definitions

Definition (Action)

A (left) ***action*** of a Lie group G on a manifold M is a smooth map $\Phi: G \times M \rightarrow M$, written $\Phi_g(m) = g \cdot m$, satisfying

$$e \cdot m = m, \quad g \cdot (h \cdot m) = (gh) \cdot m$$

for all $g, h \in G$ and $m \in M$. Equivalently, $g \mapsto \Phi_g$ sends G into the invertible transformations of M and respects products.

Lie group action

Examples

- On *vectors*: $R \cdot v = Rv$ for $v \in \mathbb{R}^3$.
- On *itself (by left multiplication)*: $g \cdot R = gR$ for $R \in \text{SO}(3)$

Definition (Conjugation action)

For $g \in G$, the ***conjugation*** map is

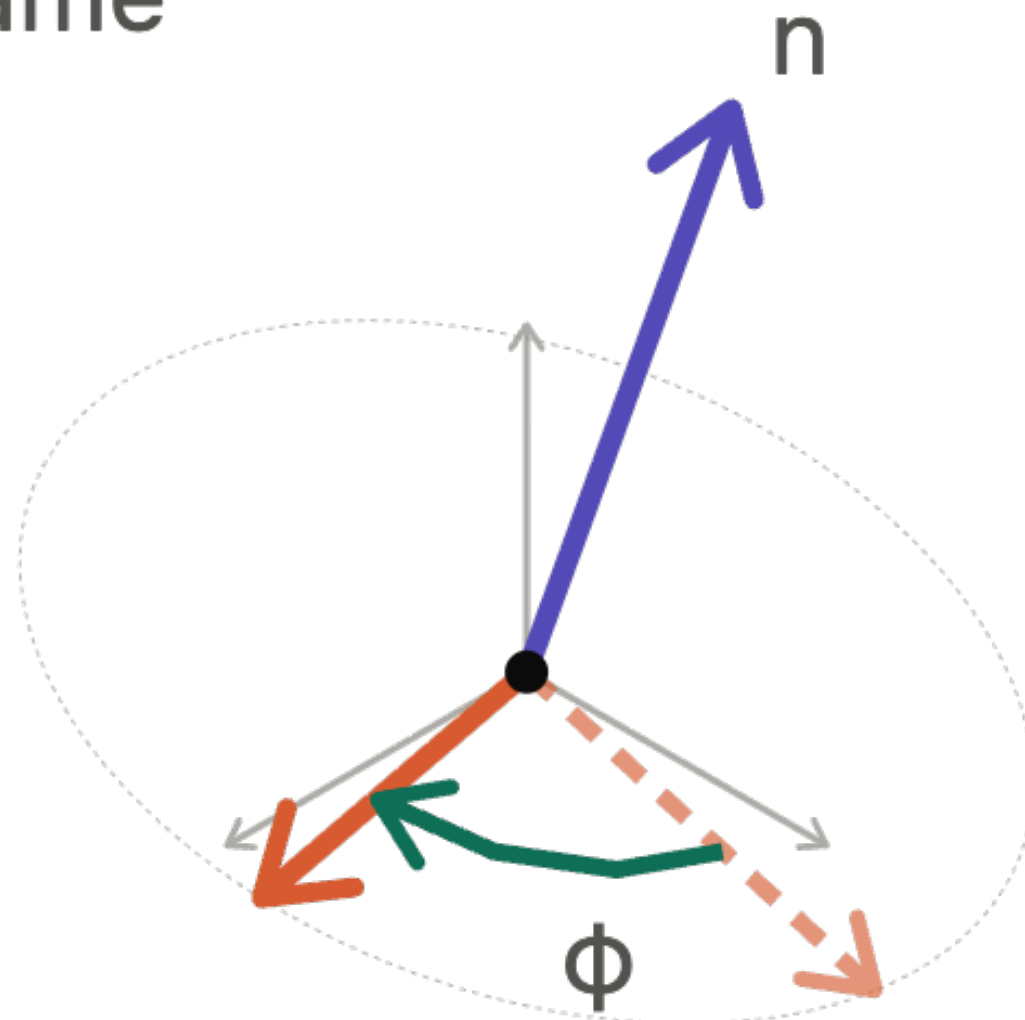
$$C_g: G \rightarrow G, \quad C_g(h) = g h g^{-1}.$$

It is a group homomorphism and fixes the identity, $C_g(e) = e$.

Lie group Conjugation action

Example

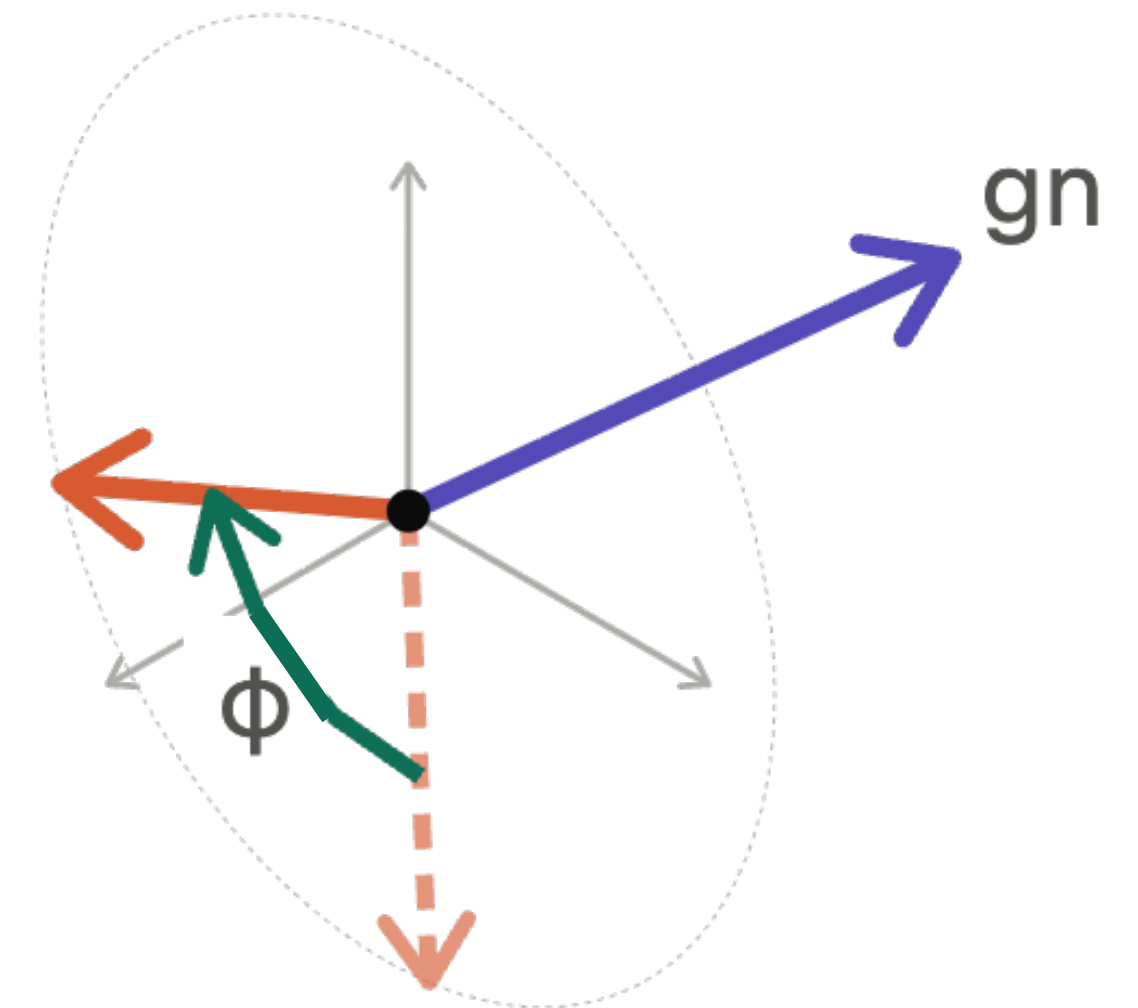
M
body frame



g



W
space frame



$C_g(h) = g h g^{-1}$: axis $n \mapsto gn$ turns, angle ϕ stays the same

Lie group Adjoint action

Definitions

Definition (Adjoint representation)

Fix g and differentiate C_g at the identity. Since $C_g(e) = e$, the derivative is a linear map on $\mathfrak{g} = T_e G$, the ***adjoint representation***

$$\text{Ad}_g = (dC_g)_e : \mathfrak{g} \rightarrow \mathfrak{g}.$$

Concretely, choose any curve $h(t)$ in G with $h(0) = e$, $\dot{h}(0) = \xi$; then

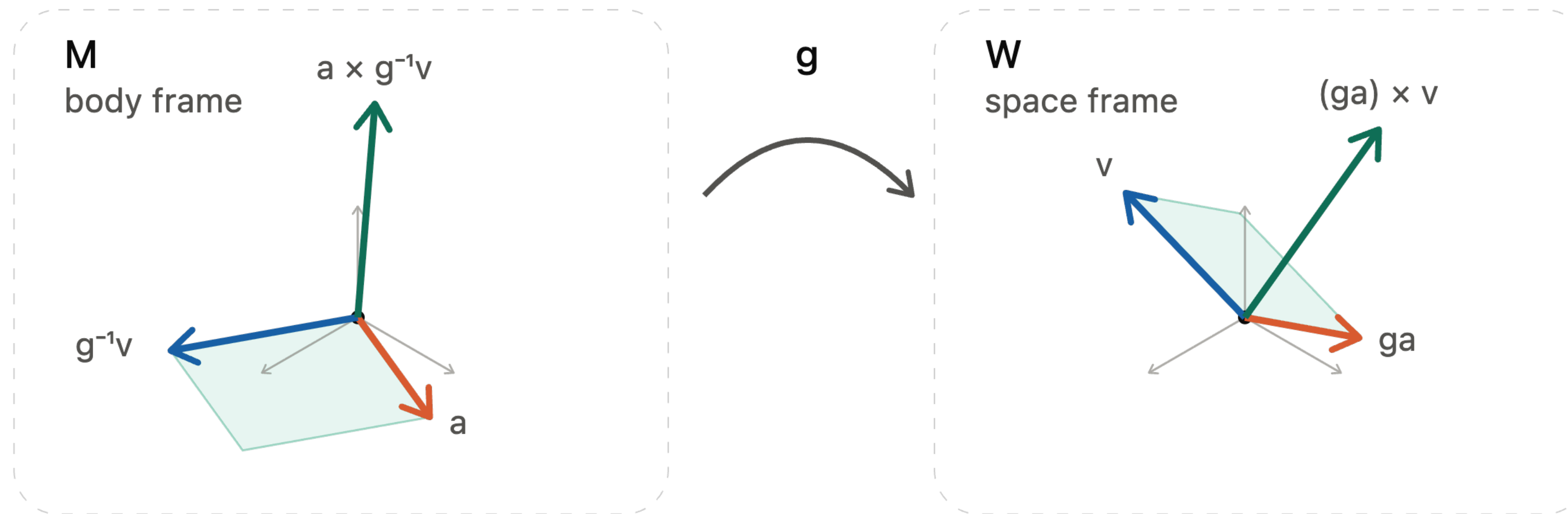
$$\text{Ad}_g \xi = \left. \frac{d}{dt} \right|_0 g h(t) g^{-1}.$$

This carries an infinitesimal generator from one frame into another.

Lie group Adjoint action

Examples

- The hat map $\hat{\cdot} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$, gives $\hat{a} b = a \times b$.
- Because a rotation carries cross products along, $g(a \times v) = (ga) \times (gv)$
- Then $g \hat{a} g^{-1} v = g(a \times (g^{-1}v)) = (ga) \times v$
- Giving $\text{Ad}_g \hat{a} = \widehat{ga}$



g rotates the whole tripod: $g(a \times g^{-1}v) = (ga) \times v$

Lie group Lie bracket action

Definitions

Definition (Infinitesimal adjoint action)

Differentiate Ad a second time, now in g along $g(t) = \exp(t\zeta)$.

The result is a linear map on $\mathfrak{g}$, equal to the Lie bracket:

$$\text{ad}_\zeta : \mathfrak{g} \rightarrow \mathfrak{g}, \quad \text{ad}_\zeta \xi = \left. \frac{d}{dt} \right|_0 \text{Ad}_{\exp(t\zeta)} \xi$$

■ **Example 2.9** — ad for $\text{SO}(3)$ is the cross product. The hat map sends the commutator to the cross product (the identity from the $\mathfrak{so}(3)$ box above),

$$\text{ad}_{\hat{a}} \hat{b} = [\hat{a}, \hat{b}] = \widehat{a \times b}, \quad \text{i.e.} \quad \text{ad}_a b = a \times b.$$

Lie group coadjoint action

Definitions

Definition (Coadjoint actions)

Momenta live in the dual $\mathfrak{g}^*$, so we need the duals of Ad_g and ad_ζ . They are defined by the pairing $\langle \cdot \mid \cdot \rangle$ between $\mathfrak{g}^*$ and $\mathfrak{g}$:

$$\langle \text{Ad}_g^* \mu \mid \xi \rangle = \langle \mu \mid \text{Ad}_g \xi \rangle, \quad \langle \text{ad}_\zeta^* \mu \mid \xi \rangle = \langle \mu \mid \text{ad}_\zeta \xi \rangle,$$

for all $\xi, \eta \in \mathfrak{g}$, $\mu \in \mathfrak{g}^*$. The **coadjoint action** of G on $\mathfrak{g}^*$ is $g \mapsto \text{Ad}_{g^{-1}}^*$ (the inverse makes it a left action).

Lie group coadjoint action

Example

■ **Example 2.10 — Coadjoint maps for $SO(3)$.** Use the trace pairing $\langle \xi \mid \eta \rangle = -\frac{1}{2} \text{tr}(\xi\eta)$, which under the hat map is the dot product, $\langle \hat{a} \mid \hat{b} \rangle = a \cdot b$, and identifies $\mathfrak{so}(3)^* \cong \mathbb{R}^3$. The two maps then unwind into vector calculus. From $\langle \text{Ad}_g^* m \mid a \rangle = \langle m \mid ga \rangle = (g^{-1}m) \cdot a$,

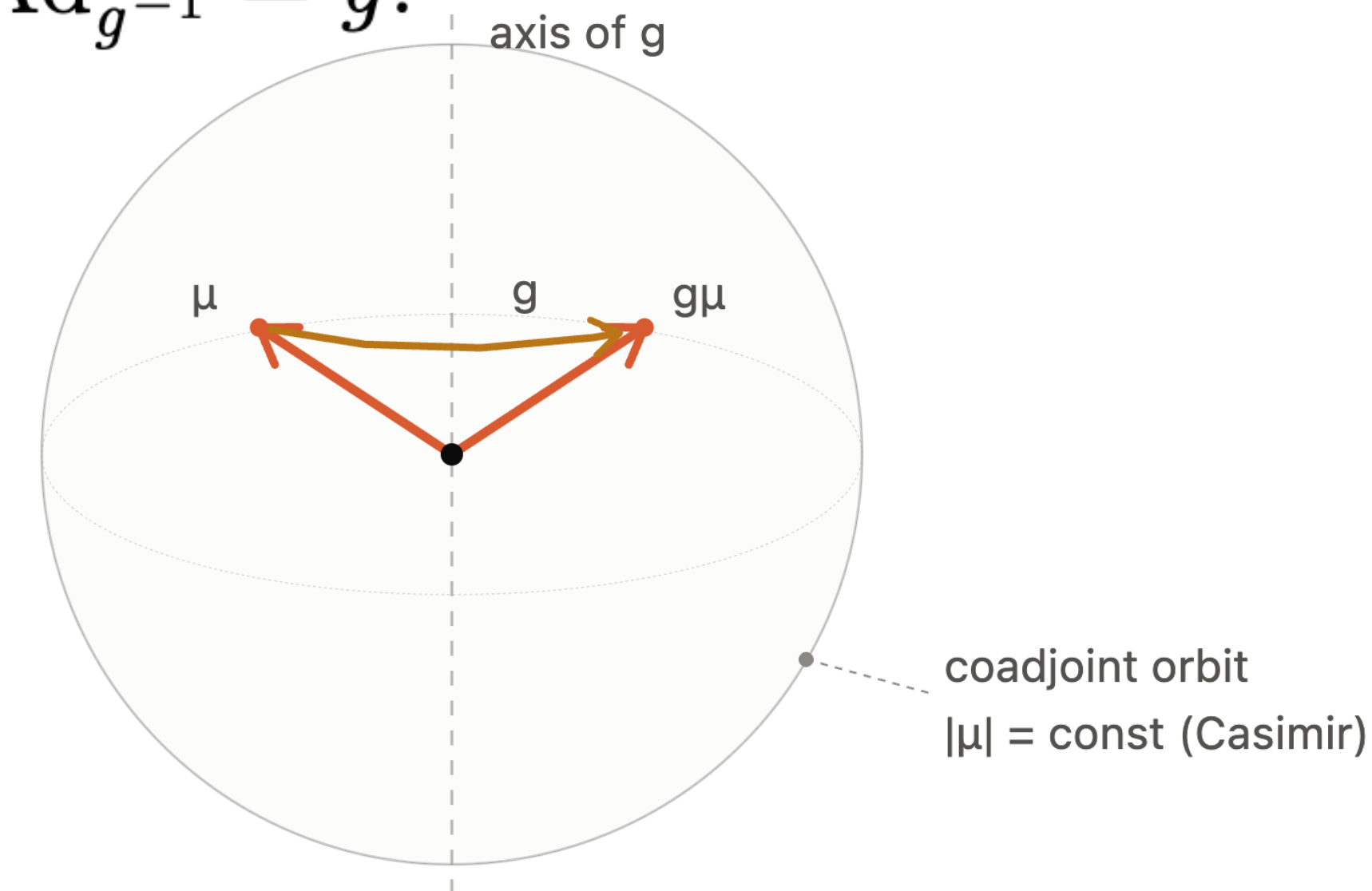
$$\text{Ad}_g^* = g^{-1}, \quad \text{so the coadjoint action is} \quad \text{Ad}_{g^{-1}}^* = g.$$

$$\text{From } \langle \text{ad}_{\hat{a}}^* m \mid b \rangle = \langle m \mid a \times b \rangle = (m \times a) \cdot b,$$

$$\text{ad}_{\hat{a}}^* m = m \times a.$$

Euler's equation for rigid bodies

$$\frac{d}{dt}(\mathbb{I}\Omega) = \mathbb{I}\Omega \times \Omega$$



$\text{Ad}_{g^{-1}}^* = g$ rotates the momentum μ on its sphere

Lie group symmetry

Definition

Definition (Symmetry)

Let M carry a structure, e.g., a function, a metric, or a symplectic form. An action of G on M is a ***symmetry*** of that structure if every Φ_g preserves it.

- For an energy $E: M \rightarrow \mathbb{R}$: $E(g \cdot m) = E(m)$ for all g, m ;
- For a Riemannian metric: each Φ_g is an isometry;
- For a symplectic form ω : $\Phi_g^* \omega = \omega$, i.e., each Φ_g is a symplectomorphism.

Structures in PDEs Modelling Nature

Noether's theorem

- For every continuous symmetry of system $\leftrightarrow$ an equivalent conservation law.
- Time translation symmetry $\longleftrightarrow$ Conservation of energy
- Translation symmetry $\longleftrightarrow$ Conservation of linear momentum
- Rotation symmetry $\longleftrightarrow$ Conservation of angular momentum
- Gauge symmetry $\longleftrightarrow$ e.g. Conservation of electric charge
- Particle relabelling symmetry $\longleftrightarrow$ Conservation of circulation



Emmy Noether

Lie group reduction due to symmetry

Special change of variables that preserve the structure

- Since the Lagrangian is invariant under these symmetry directions, some configurations are redundant degrees of freedom.

$$L(q, \dot{q}) = L(\Phi_g(q), \frac{d}{dt} \Phi_g(q))$$

- So, we quotient out the redundant directions (reduction), and express the Lagrangian in reduced variables.

$$L(q, \dot{q}) \xrightarrow{\text{reduce}} \ell(u)$$

- Reduced action based on reduced Lagrangian

$$s(\Omega) = \int_0^T \ell(\Omega) dt$$

Reduced Variational Optimization

After change of variables they get Lin constraints

Definition (Lin constraints)

Write $\eta = R^{-1} \delta R$ for the variation of $R(t)$ translated to the identity and vanishing at the endpoints.

$$\delta\Omega = \dot{\eta} + [\Omega, \eta] = \dot{\eta} + \text{ad}_{\Omega} \eta, \quad \eta(t_0) = \eta(t_1) = 0,$$

This relation on is called the ***Lin constraints***.

Reduced Variational Optimization

After change of variables they get Lin constraints

Reduced action based on reduced Lagrangian

$$s(\Omega) = \int_0^T \ell(\Omega) dt$$

Reduced Hamilton's stationary action principle

$$\frac{\delta s(\Omega)}{\delta \Omega} = 0 \quad \text{s.t} \quad \delta \Omega = \dot{\eta} + \text{ad}_{\Omega} \eta$$

Reduced Hamilton's stationary action principle

$$\frac{\partial}{\partial t} \boxed{\frac{\delta \ell}{\delta \Omega}} = \text{ad}_{\Omega}^* \boxed{\frac{\delta \ell}{\delta \Omega}} \quad \boxed{\Pi := \frac{\delta \ell}{\delta \Omega}} \quad \boxed{\dot{\Pi} = -\text{sgrad } h(\Pi)}$$

Conjugate momentum

Symplectic foliation

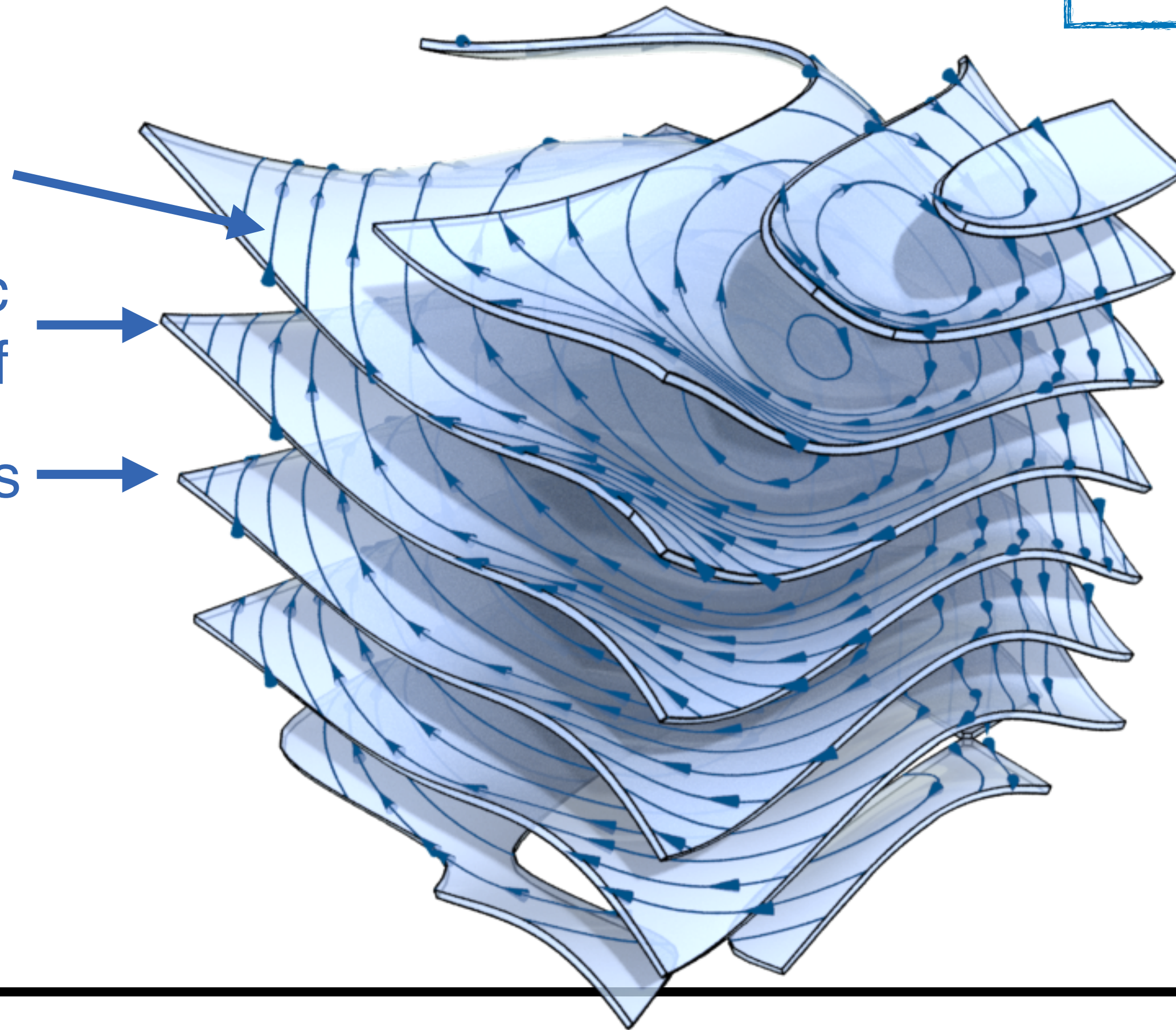
$$\Pi := \frac{\delta \ell}{\delta \Omega} \quad \left\langle \dot{\Pi} = -\text{sgrad } h(\Pi) \right.$$

Conjugate momentum

sgrad of some
function

symplectic
leaf

Coadjoint orbits



Dual Lie algebra $\mathfrak{g}^$
is a Poisson space*

Definition (Casimir)

Casimirs* or *Lie's distinguished functions are functions that are constant along each symplectic leaf.

Different Lie groups, different coadjoint actions

The setup remains the same

	Rigid body	Ideal fluid
Group G	$\mathrm{SO}(3)$	$\mathrm{Diff}_{\mathrm{vol}}(M)$
Configuration	orientation R	flow map ϕ
Frame map	$R: \mathcal{M} \rightarrow \mathcal{W}$	$\phi: \mathcal{M} \rightarrow \mathcal{W}$
Algebra $\mathfrak{g}$	$\mathfrak{so}(3)$ (skew matrices)	div-free fields $\mathfrak{X}_{\mathrm{div}}$
Ad_g	rotation g	pushforward ϕ_*
ad_ξ	cross product $\Omega \times (\cdot)$	field bracket $[u, \cdot]$
Dual $\mathfrak{g}^*$	$\mathbb{R}^3$, momentum Π	1-form densities, momentum η
Ad_g^*	rotation	pullback ϕ^*
ad_ξ^*	$\Pi \times \Omega$	Lie derivative $\mathcal{L}_u$

Rigid body dynamics

$$\ell : \mathfrak{g} \rightarrow \mathbb{R}, \qquad \ell(\Omega) = \tfrac{1}{2} \, \Omega^\top \mathbb{I} \, \Omega.$$

$$\dot{\Pi} = \Pi \times \Omega \quad \text{or} \quad \mathbb{I} \dot{\Omega} = \mathbb{I} \Omega \times \Omega$$

Ideal fluid dynamics

$$\ell : \mathfrak{g} \rightarrow \mathbb{R}, \qquad \ell(u) = \tfrac{1}{2} \int_M |u|^2 \, dx.$$

$$\frac{\partial}{\partial t} \frac{\delta \ell}{\delta u} = - \mathrm{ad}_u^* \frac{\delta \ell}{\delta u}, \quad \text{or} \quad \frac{\partial}{\partial t} \eta = - \mathcal{L}_u \eta$$

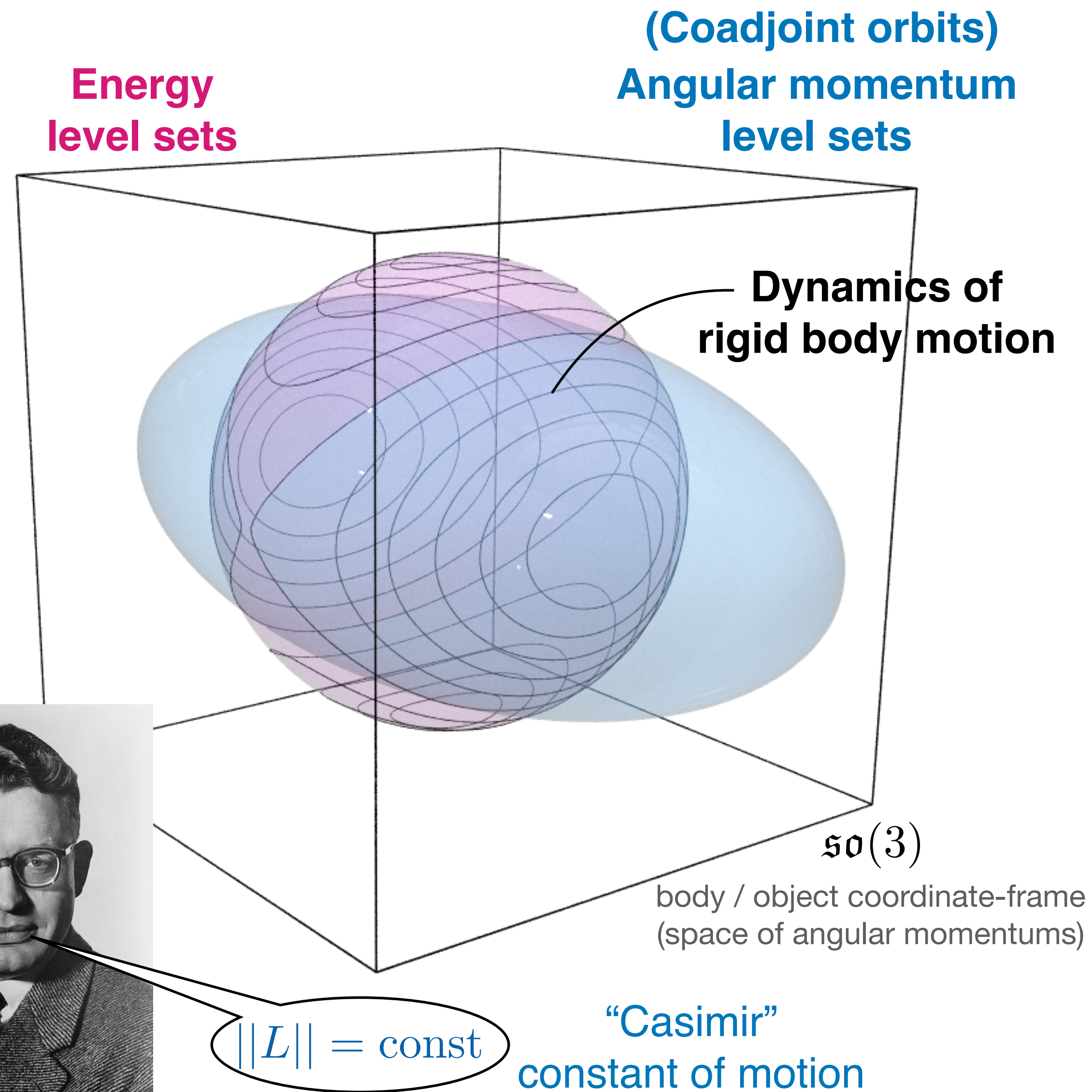
$$u = \mathbb{P}_{\mathrm{div}=0}(\eta^\sharp)$$

Rigid Body dynamics

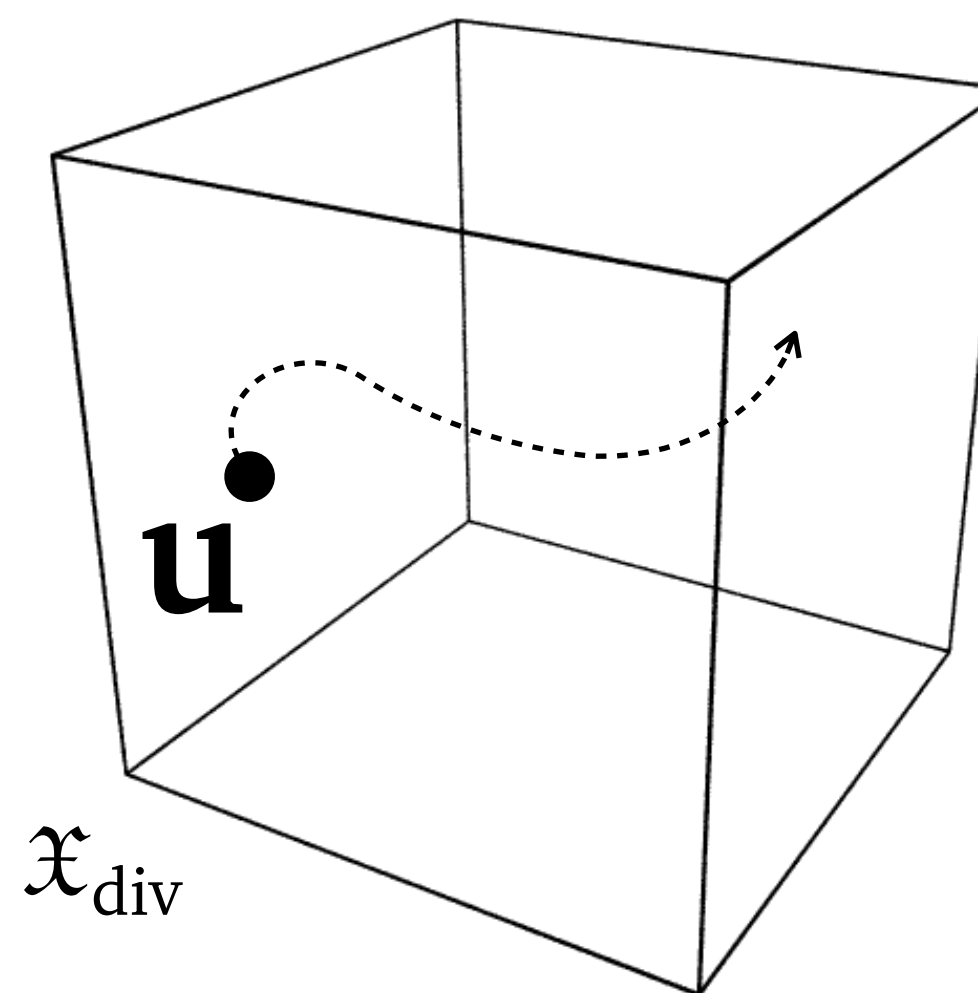
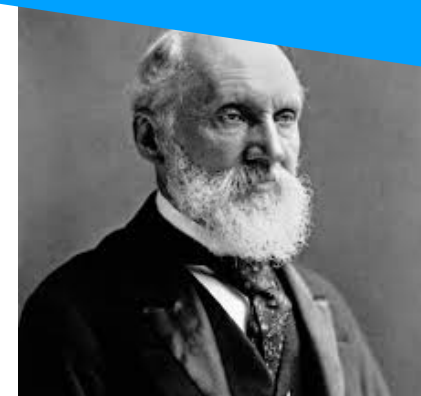
The geometric picture



Hendrik
Casimir
1909-2000

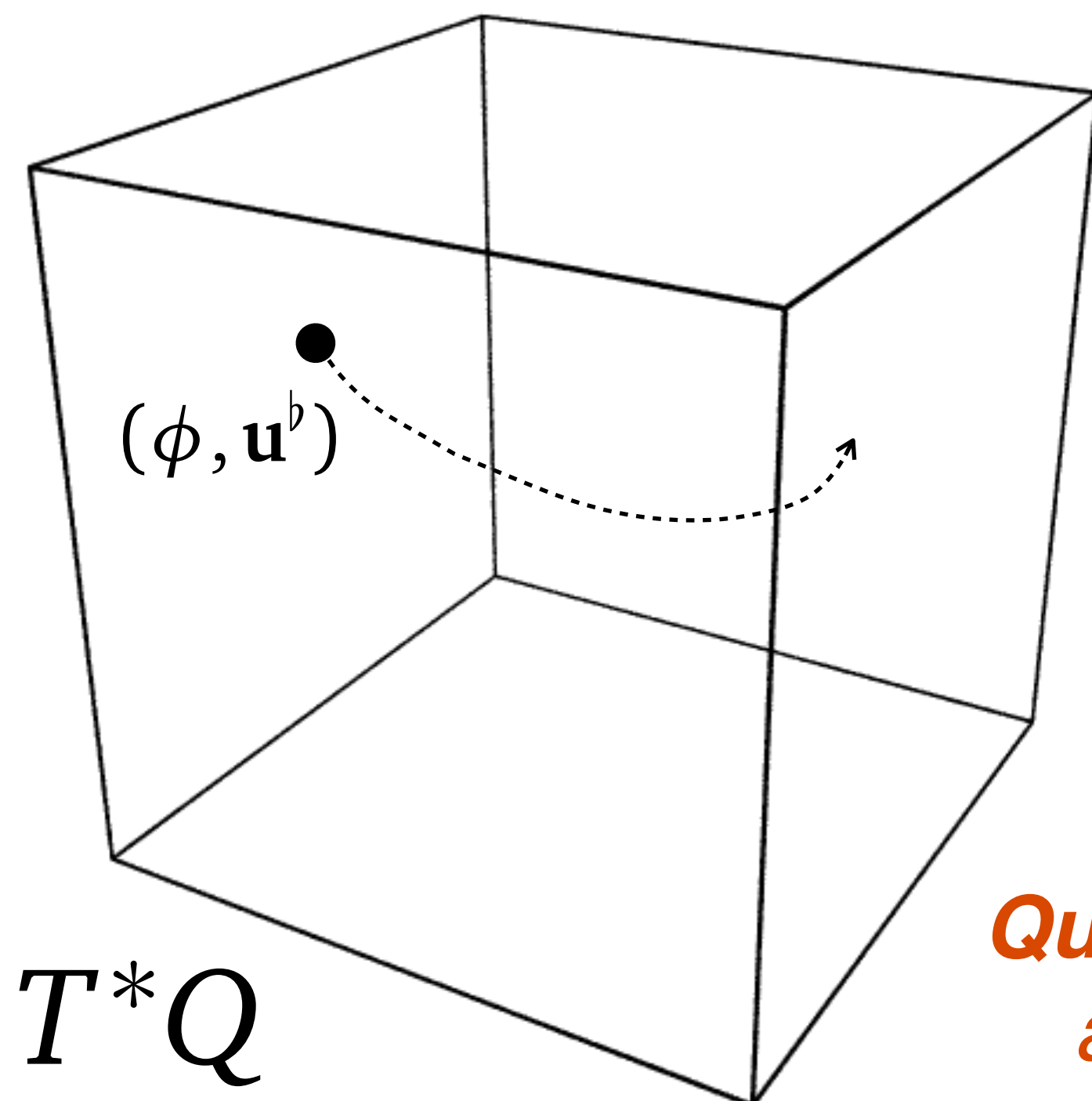


Using symmetry and conservation laws of ideal fluids



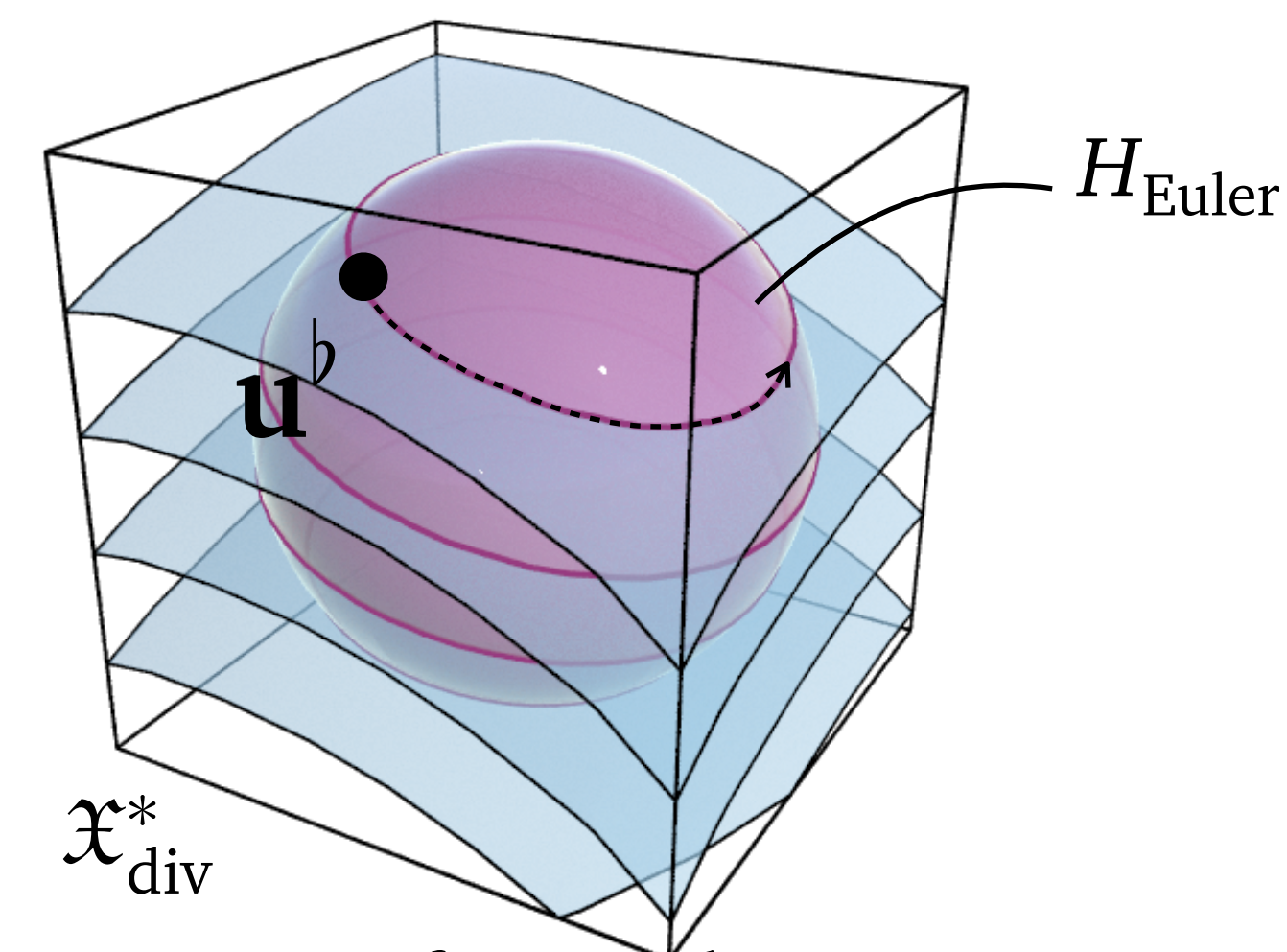
Space of divergence-free velocities

⋮ dual



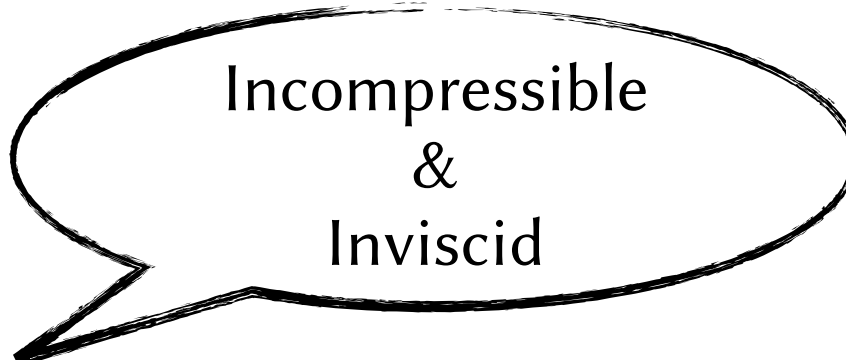
Symplectic Reduction

Momentum
map J



Space of circulations

Question: What is the quantity associated to each orbits?



Incompressible
&
Inviscid

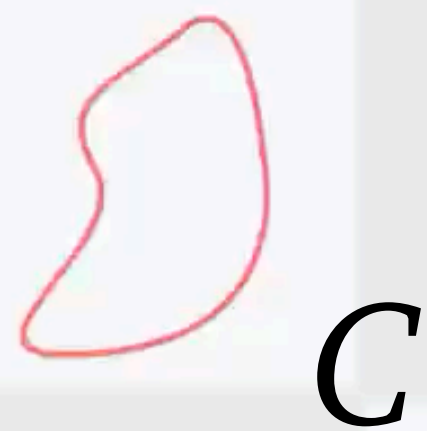
Ideal fluids dynamics

Primary objective:

Capture the intricate
vortical dynamics and
associated conservation laws.

Circulation conservation

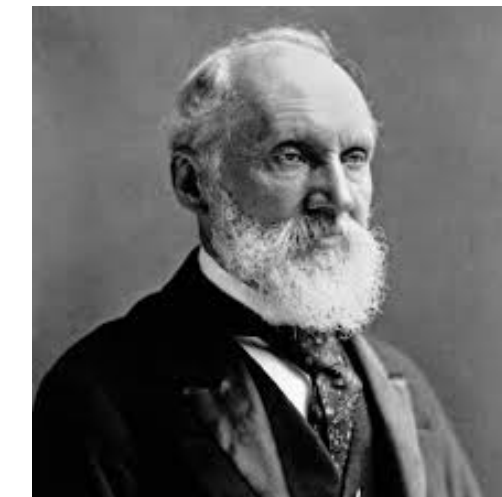
$\phi_t(C)$



Circulation: integration
of velocity covector
 $\mathbf{u}^\flat := \mathbf{u} \cdot (\cdot)$

$$\frac{d}{dt} \oint_{\phi_t(C)} \mathbf{u} \cdot d\mathbf{l} = 0 \quad \text{for all loop } C$$

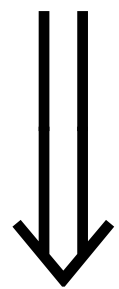
Kelvin circulation theorem



Circulation conservation

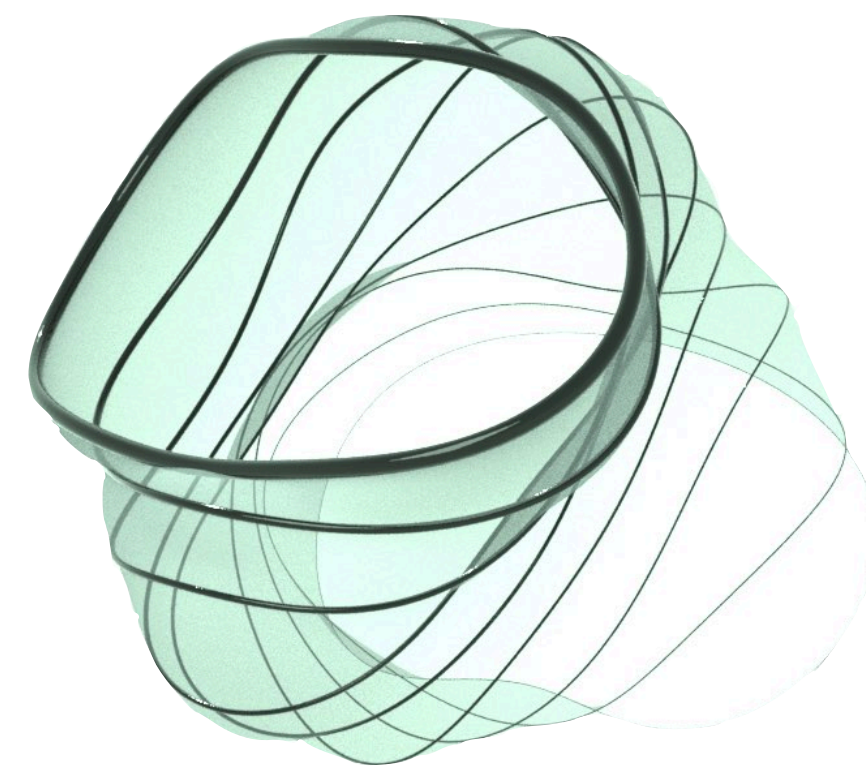
$$\frac{d}{dt} \oint_{\phi_t(C)} \mathbf{u} \cdot d\mathbf{l} = 0 \quad \text{for all loop } C$$

Kelvin circulation theorem

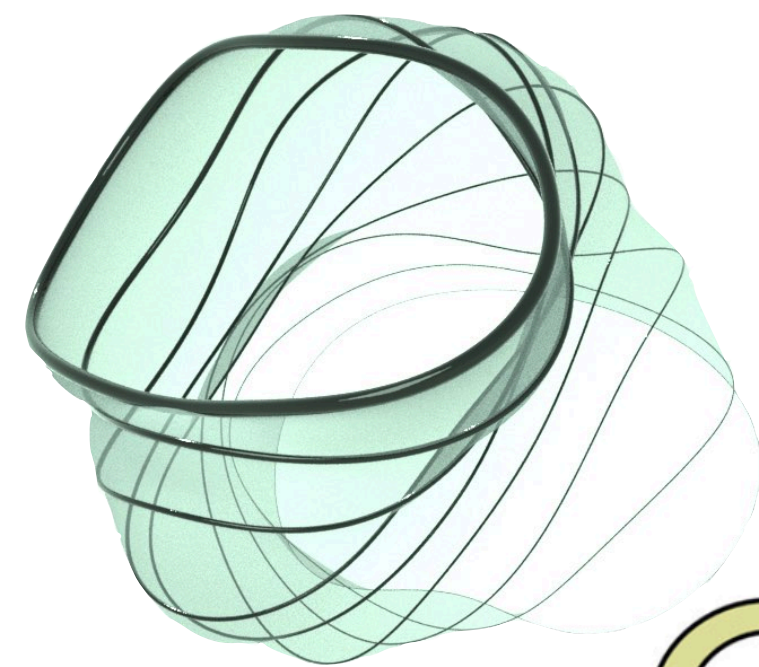


Theorem (Helmholtz 1858, Kelvin 1869)

***Vortex lines** are transported by the flow*

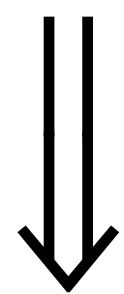


Circulation conservation



Theorem (Helmholtz 1858, Kelvin 1869)

Vortex lines are transported by the flow



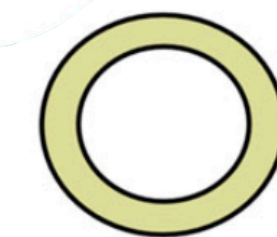
*Goal: Preserve these
Casimirs during simulation.*

2D Casimirs:

$$\mathcal{W} = \int_W f(\text{curl } \mathbf{u}) dA$$

3D Casimir (helicity):

$$\mathcal{H} = \int_W \mathbf{u} \cdot (\text{curl } \mathbf{u}) dV$$



0_1
unknot



3_1
Trefoil
knot



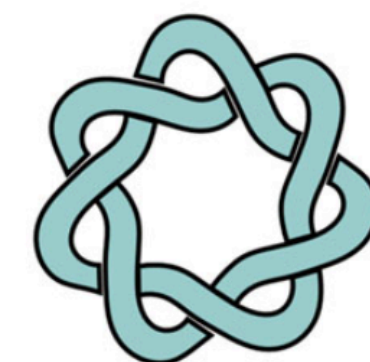
4_1
Figure-of-eight
knot



5_1
Cinquefoil
knot



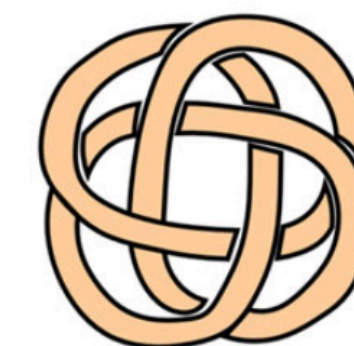
5_2
Three-twist
knot



7_1
Septofoil
knot

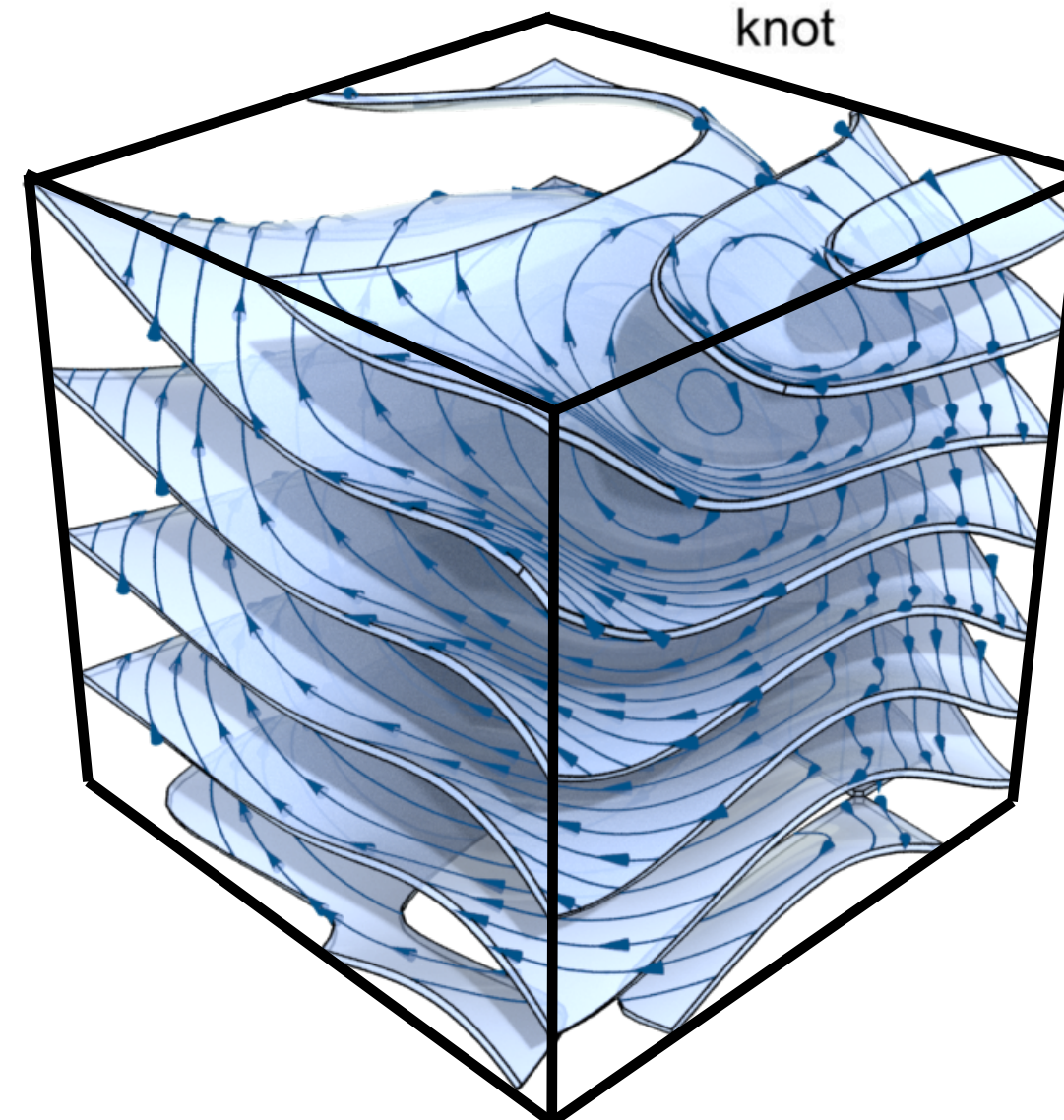


8_{18}
knot



8_{19}
knot

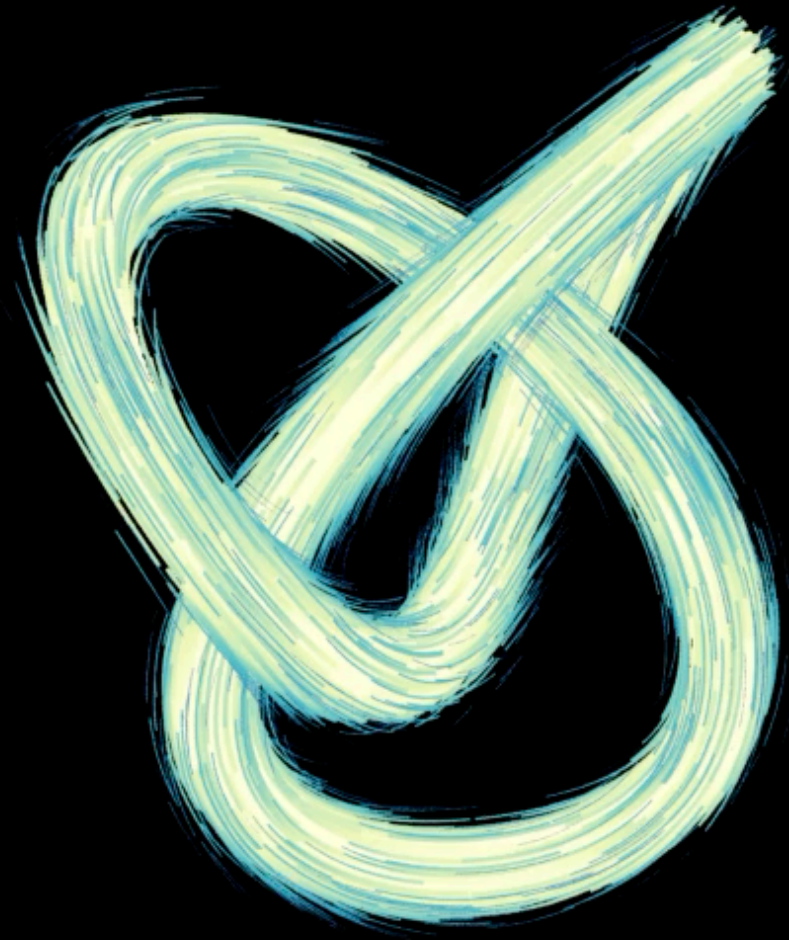
[Schaufelberger 2020]



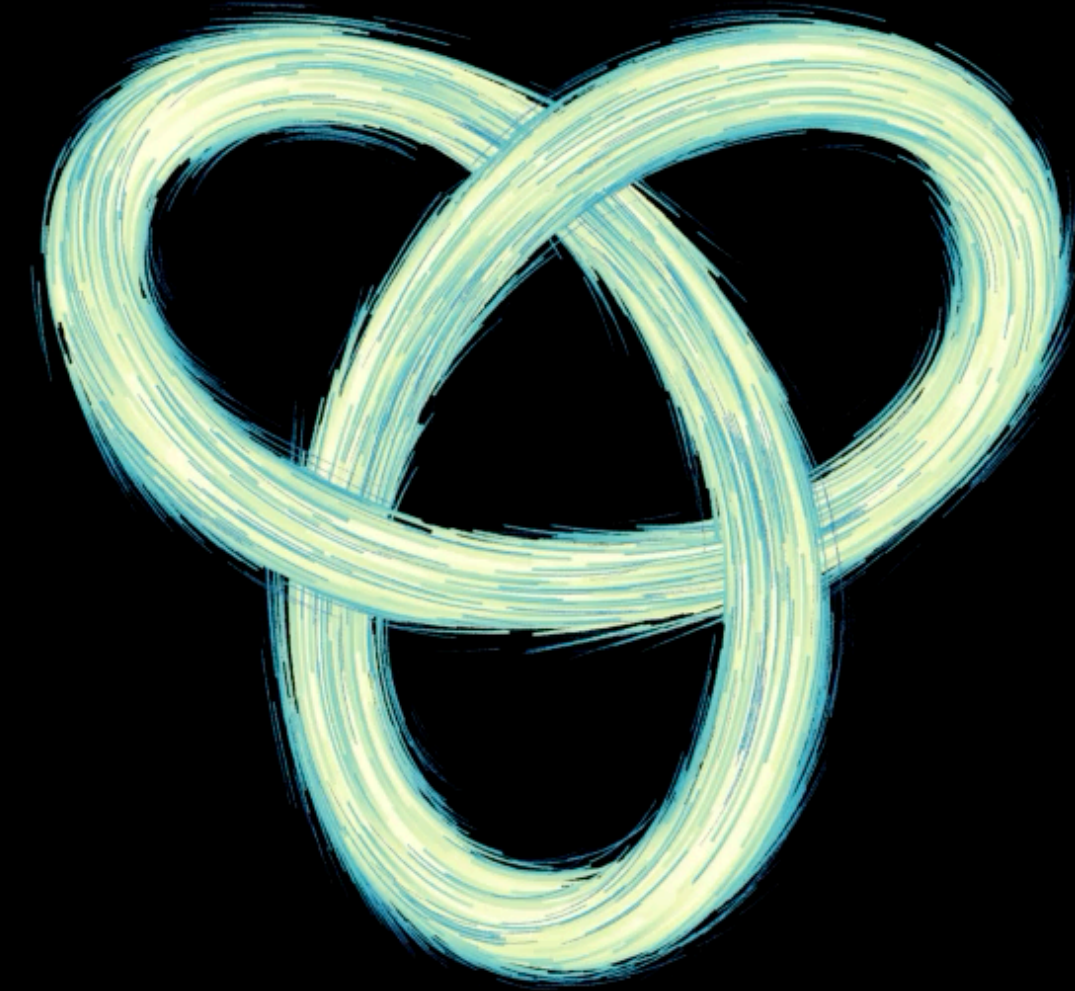
Coadjoint orbits

**Casimirs remain constant
over each orbit.**

Topological Hydrodynamics



Side view

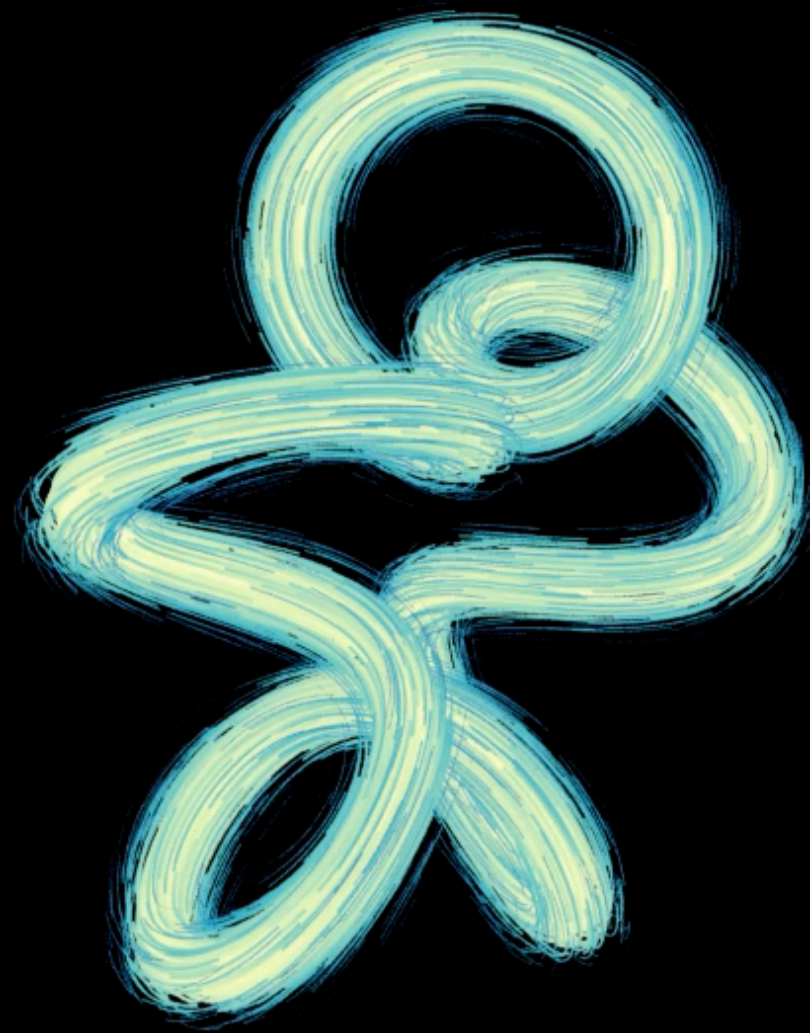


Head-on view

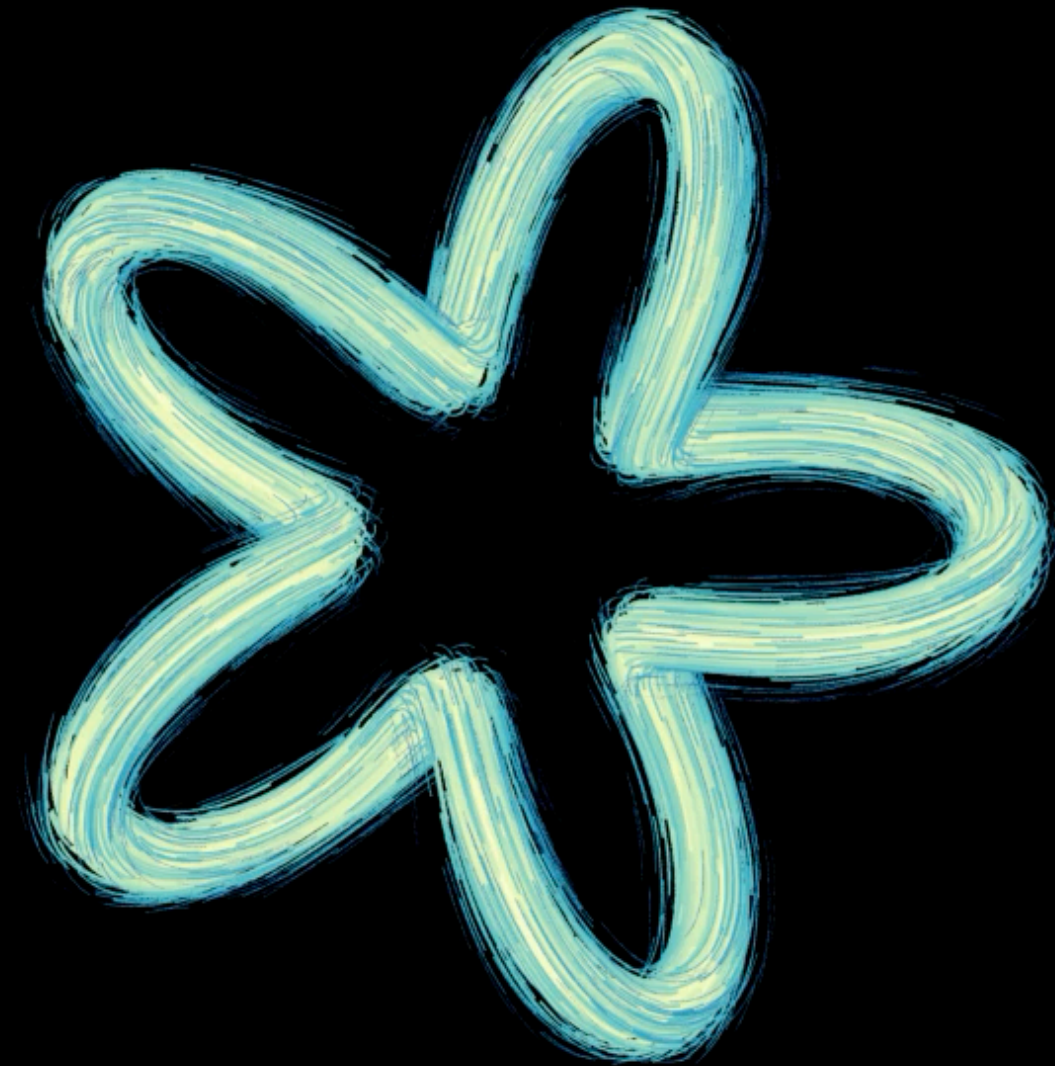
CO-FLIP (Ours)



Our method captures the unknot separation
at a low resolution of $64 \times 64 \times 64$.



Side view

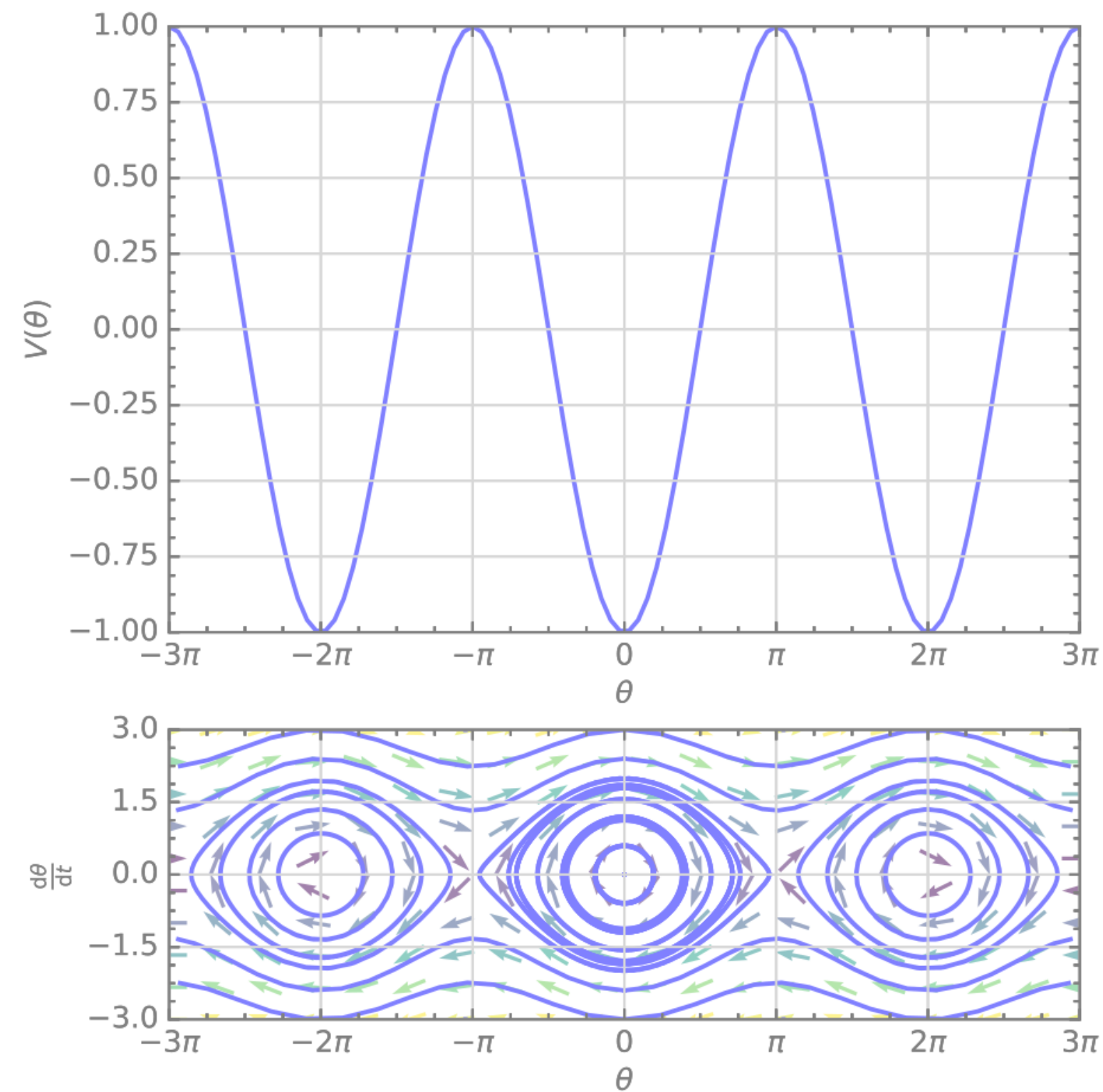


Head-on view

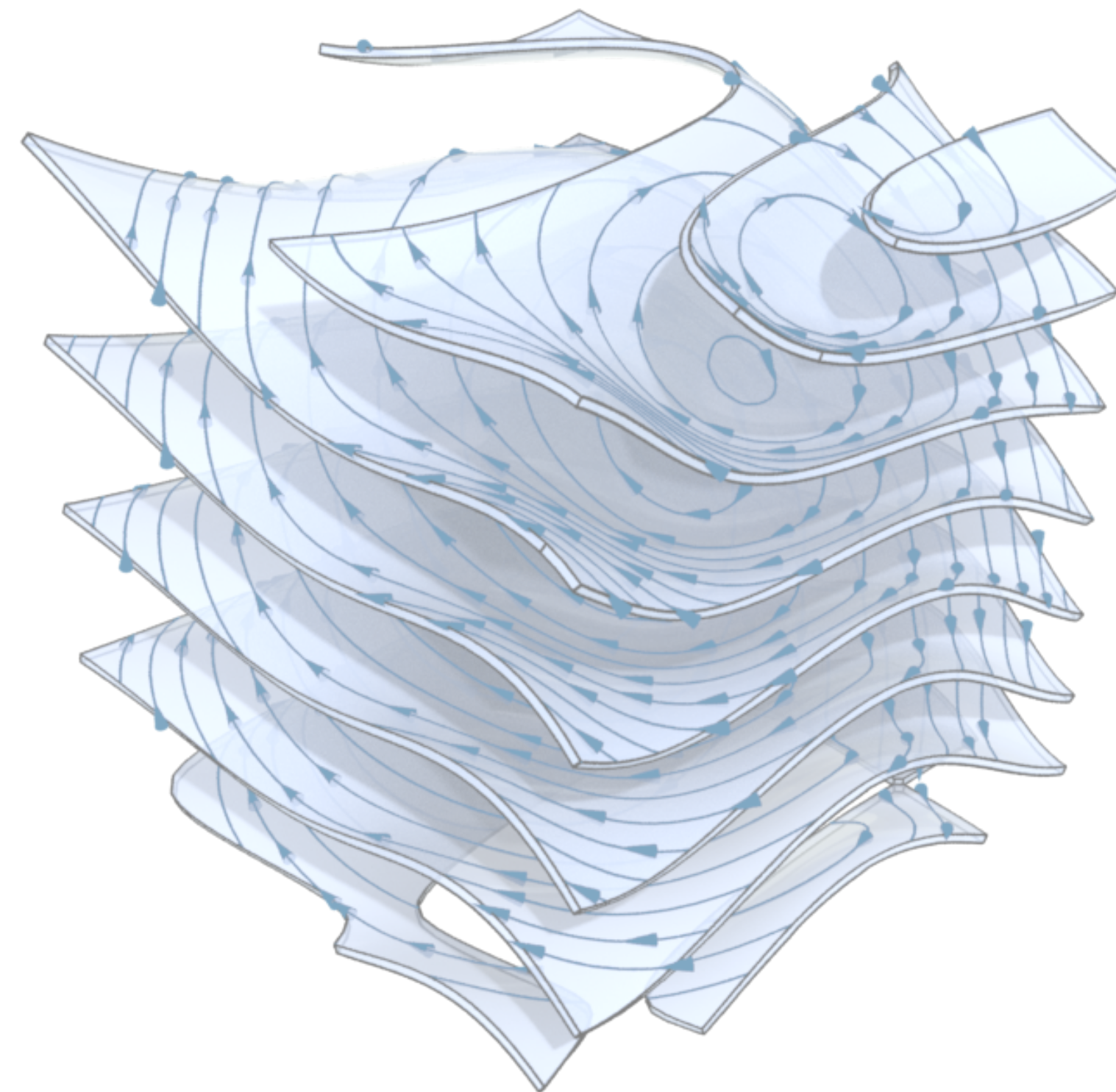
CO-FLIP (Ours)



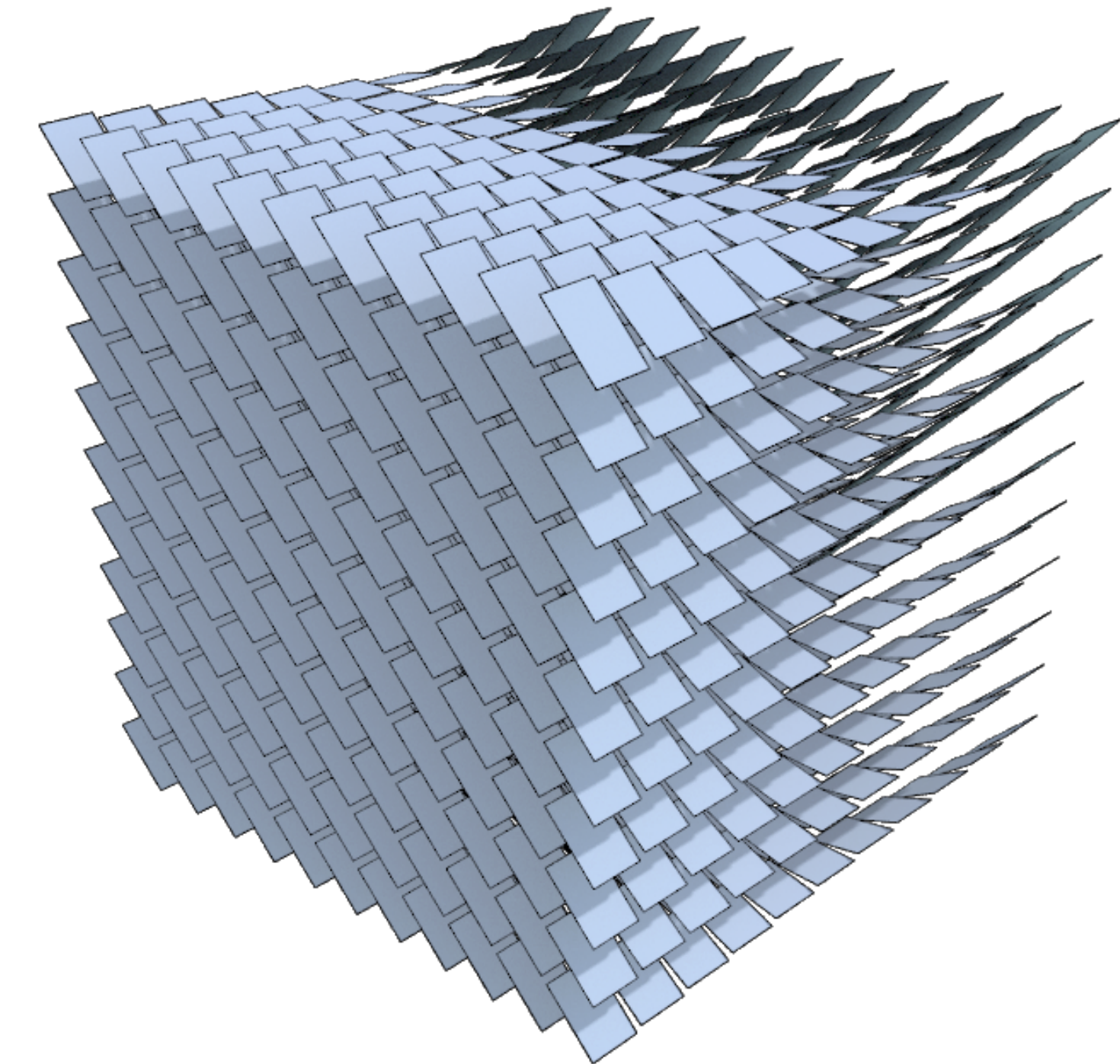
Overview



Lagrangian and
Hamiltonian Mechanics



Lie groups, Symmetry,
and Reduction



**Discretization as a
constrained variational
problem**

Applications

Holonomic systems

Systems with position constraints

$$\operatorname{argmin}_{x_t, u_t} \int_{[0,1]} \mathcal{L}(x_t, u_t) dt$$

$$\text{subject to } g(x_t) = 0$$

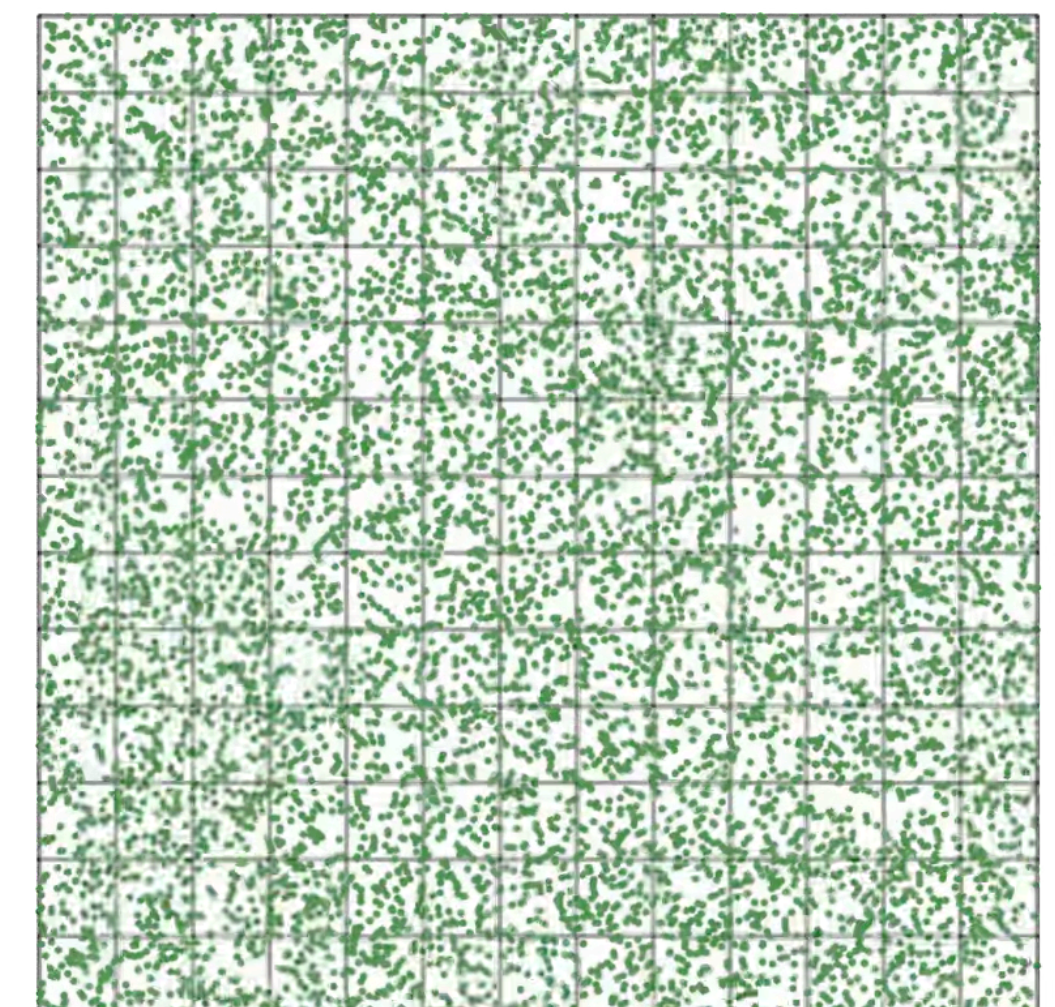
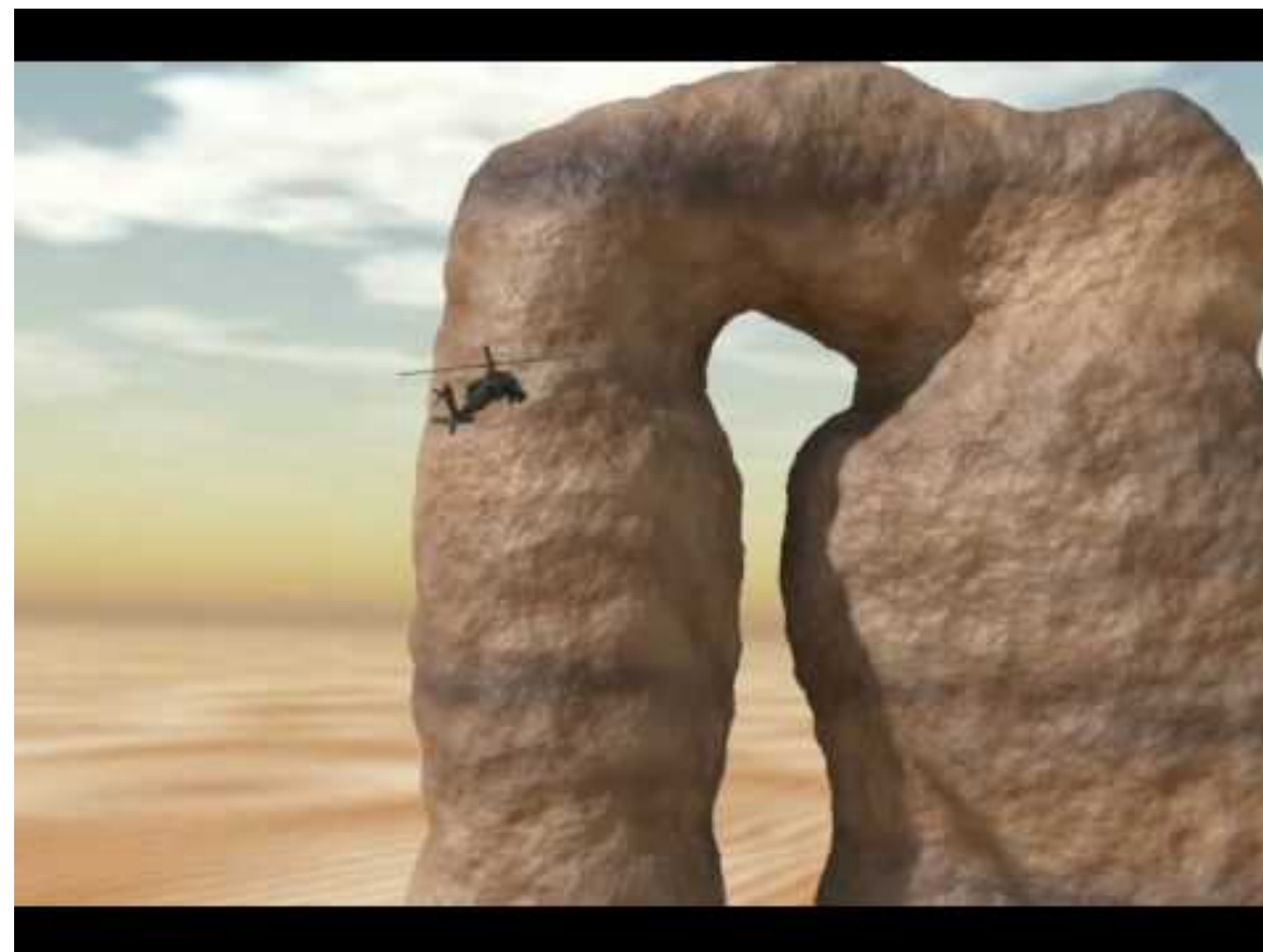
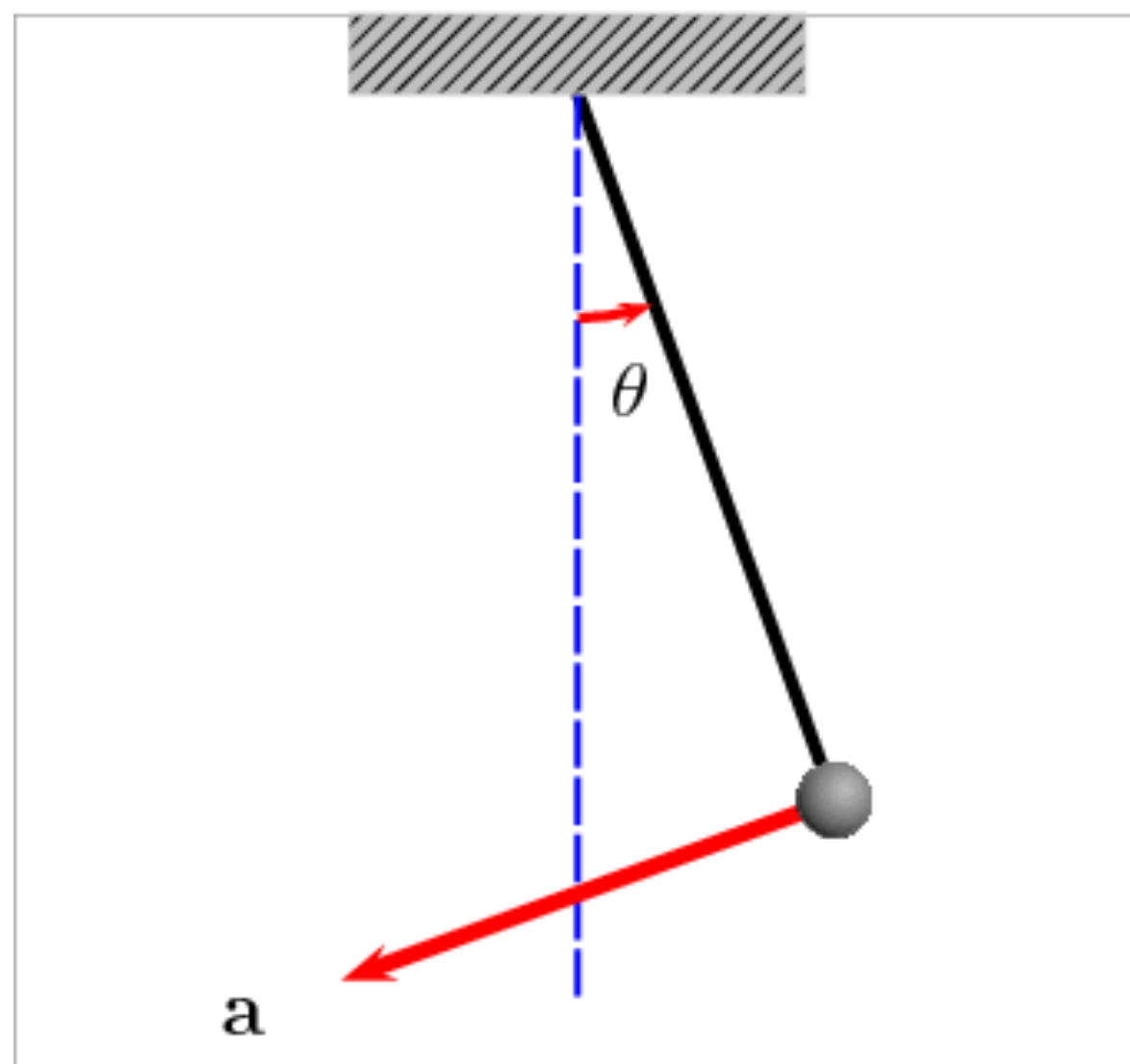
Holonomic systems

Systems with position constraints

$$\operatorname{argmin}_{x_t, u_t} \int_{[0,1]} \mathcal{L}(x_t, u_t) dt$$

$$\text{subject to } g(x_t) = 0$$

- Pendulum motion —> Constraint: Fixed wire length
- Helicopter motion —> Constraint: Path obstacles
- Ideal Fluid motion —> Constraint: Volume preservation



Nonholonomic systems

Systems with velocity constraints

$$\operatorname{argmin}_{x_t, u_t} \int_{[0,1]} \mathcal{L}(x_t, u_t) dt$$

$$\text{subject to } g(x_t) = 0$$

$$\text{subject to } h(u_t) = 0$$

Nonholonomic systems

Systems with velocity constraints

$$\operatorname{argmin}_{x_t, u_t} \int_{[0,1]} \mathcal{L}(x_t, u_t) dt$$

$$\text{subject to } g(x_t) = 0$$

$$\text{subject to } h(u_t) = 0$$

- Maneuvering cars / snakeboards $\rightarrow$ Constraints: Fixed actuation directions

Nonholonomic systems

Systems with velocity constraints

$$\operatorname{argmin}_{x_t, u_t} \int_{[0,1]} \mathcal{L}(x_t, u_t) dt$$

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- Maneuvering cars / snakeboards $\rightarrow$ Constraints: Fixed actuation directions

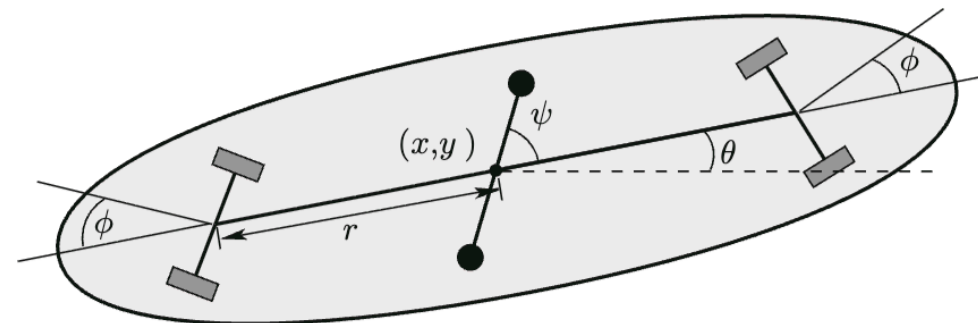


FIGURE 5.8.1. The geometry of the snakeboard.



Nonholonomic systems

Systems with velocity constraints

$$\operatorname{argmin}_{x_t, u_t} \int_{[0,1]} \mathcal{L}(x_t, u_t) dt$$

$$\text{subject to } g(x_t) = 0$$

$$\text{subject to } h(u_t) = 0$$

- Maneuvering cars / snakeboards $\rightarrow$ Constraints: Fixed actuation directions
- Rolling ball without slipping $\rightarrow$ Constraints: Angular velocity match tangential velocity

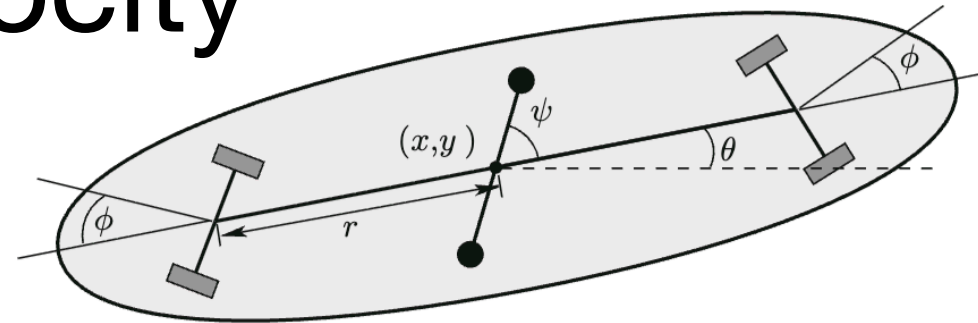
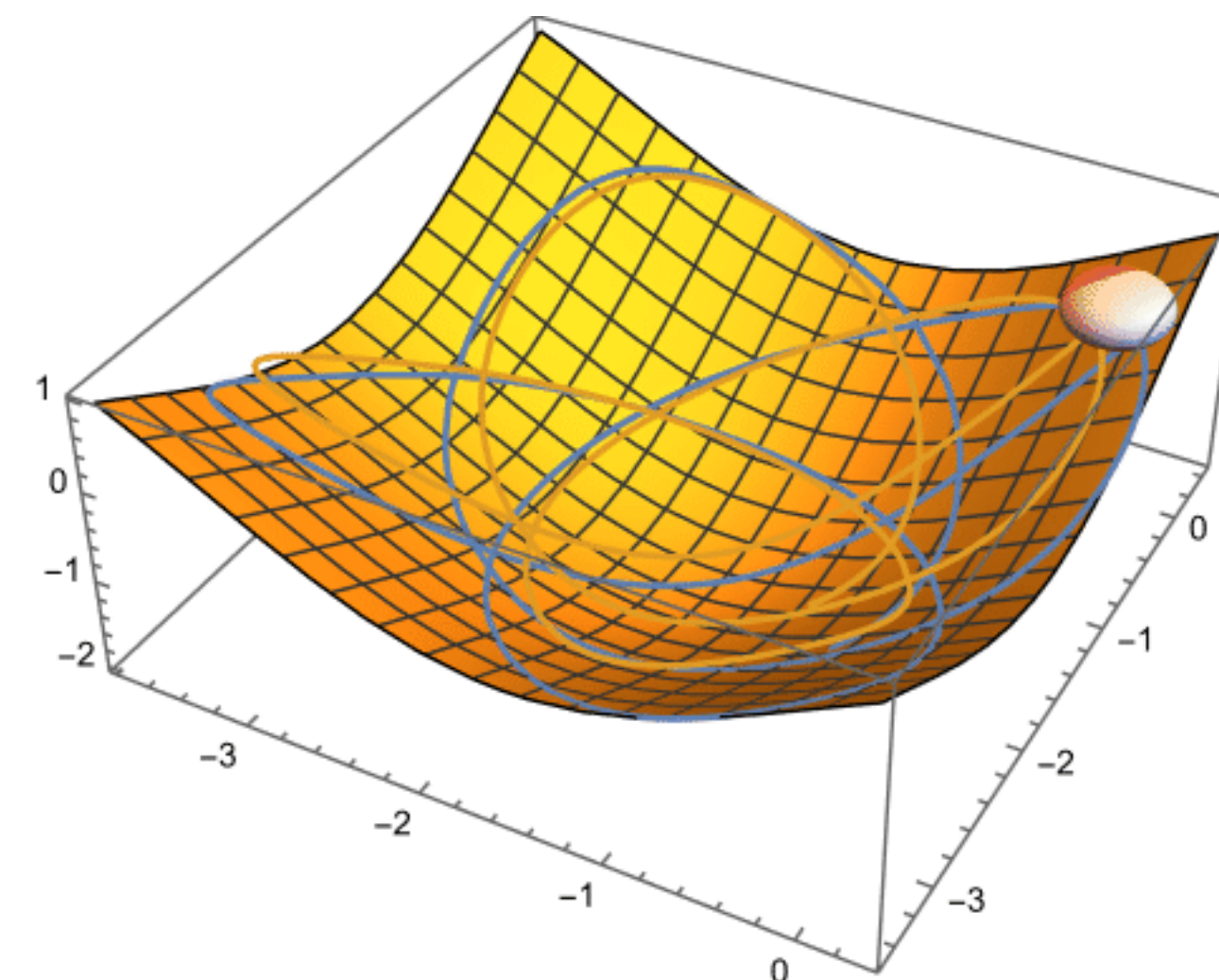


FIGURE 5.8.1. The geometry of the snakeboard.



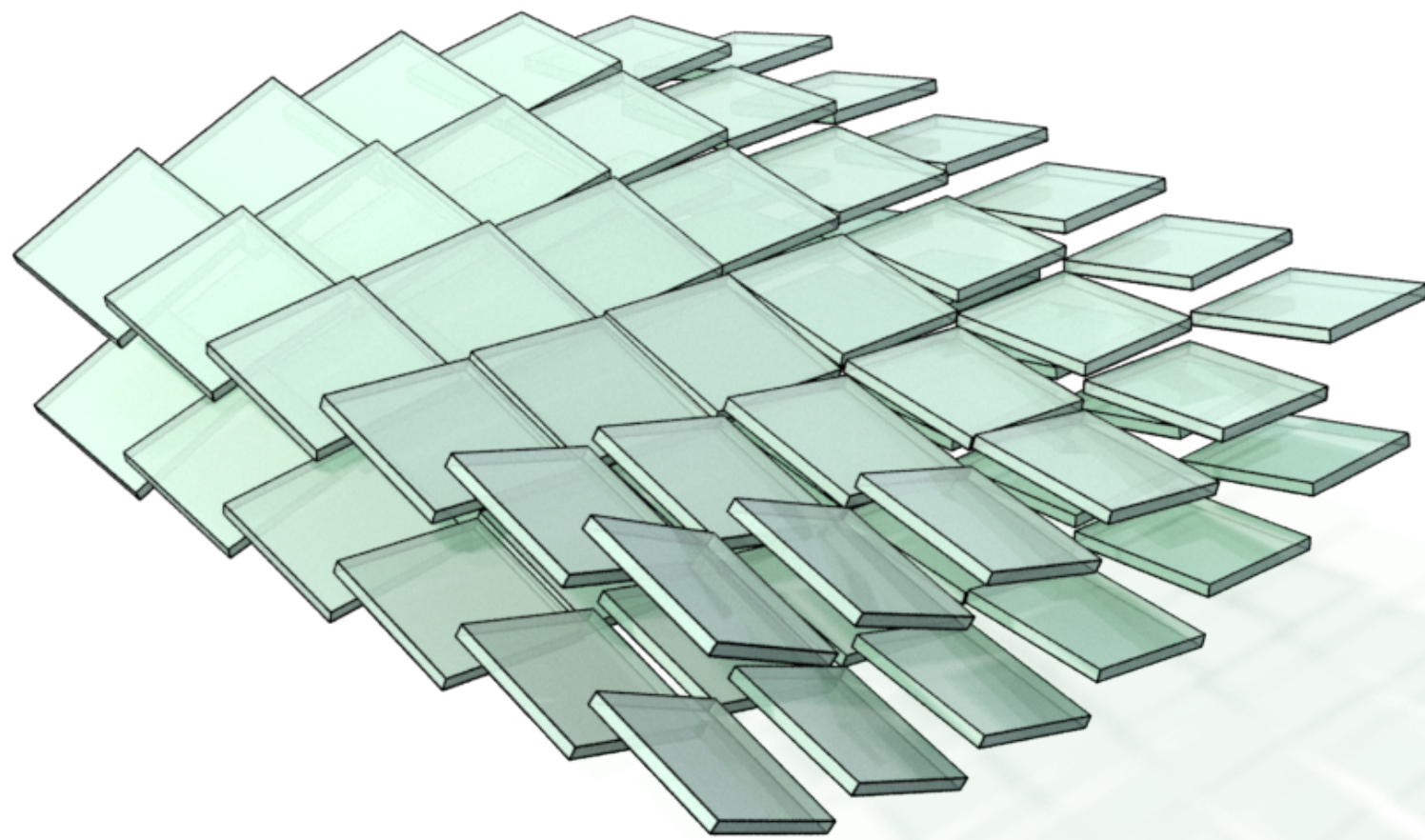
Recap of a constrained physical world

Nonholonomic and holonomic constraints

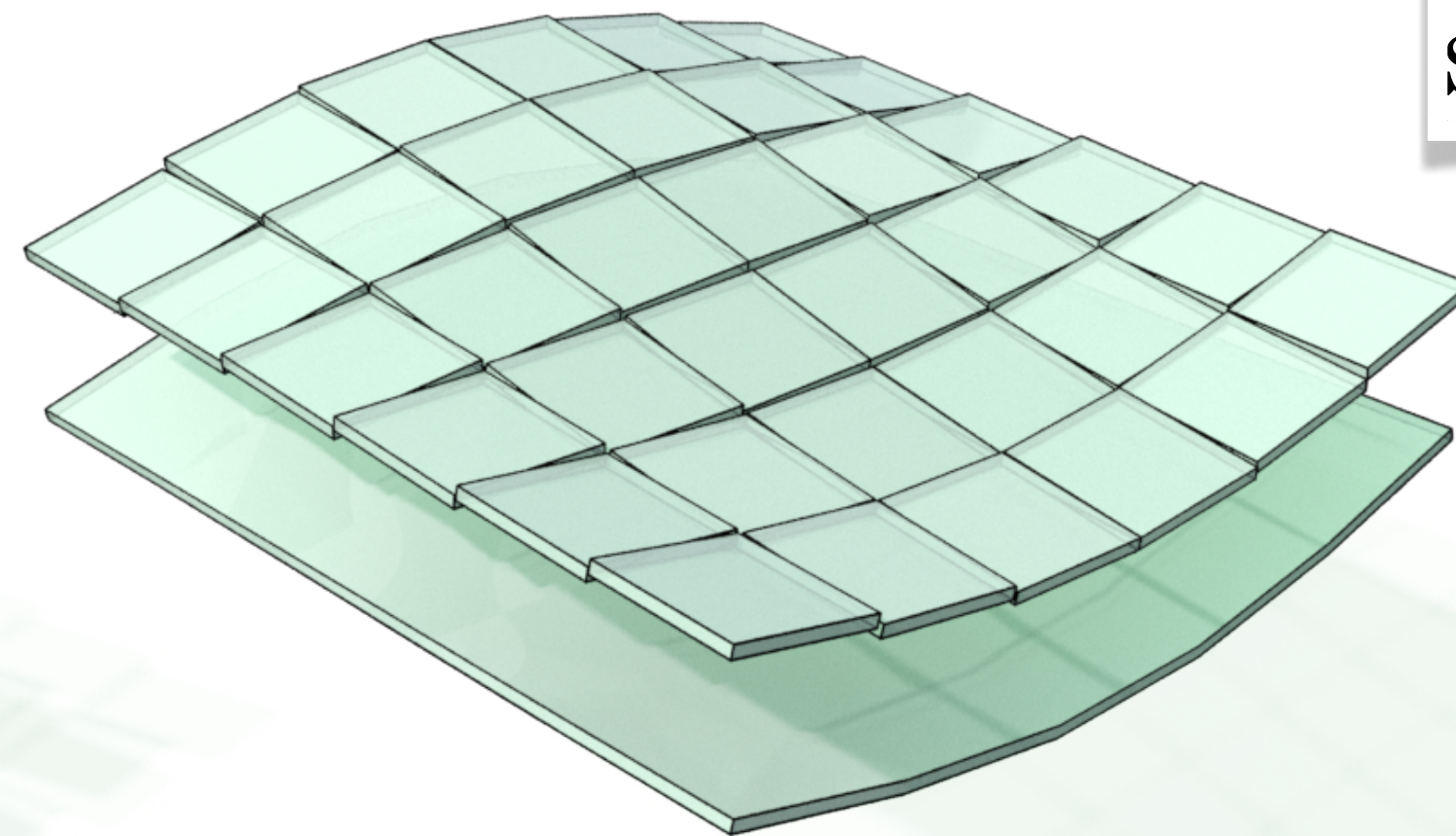
$$\operatorname{argmin}_{x_t, u_t} \int_{[0,1]} \mathcal{L}(x_t, u_t) dt$$

$$\text{subject to } g(x_t) = 0$$

$$\text{subject to } h(u_t) = 0$$

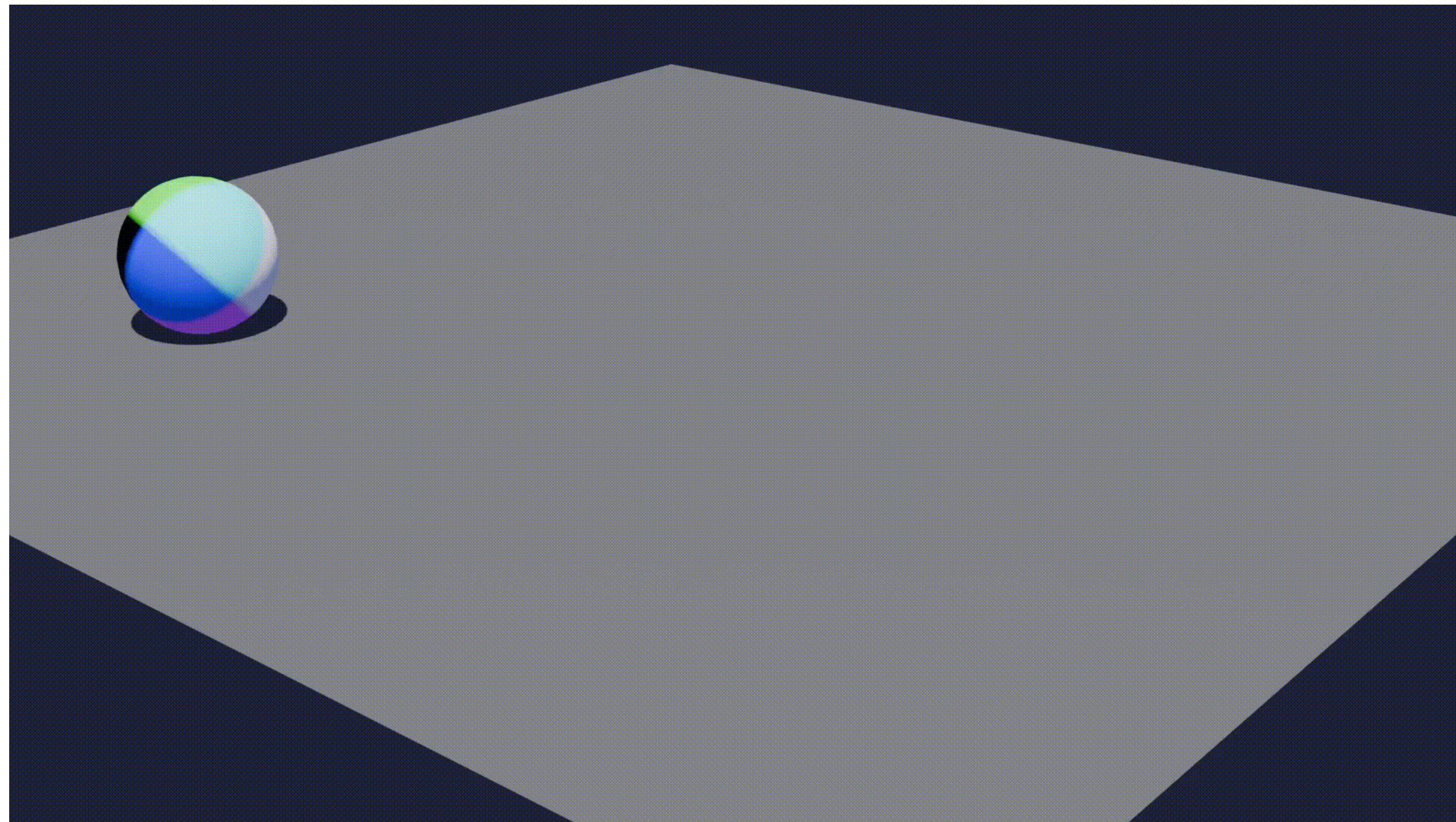


Non-holonomic / Non-integrable



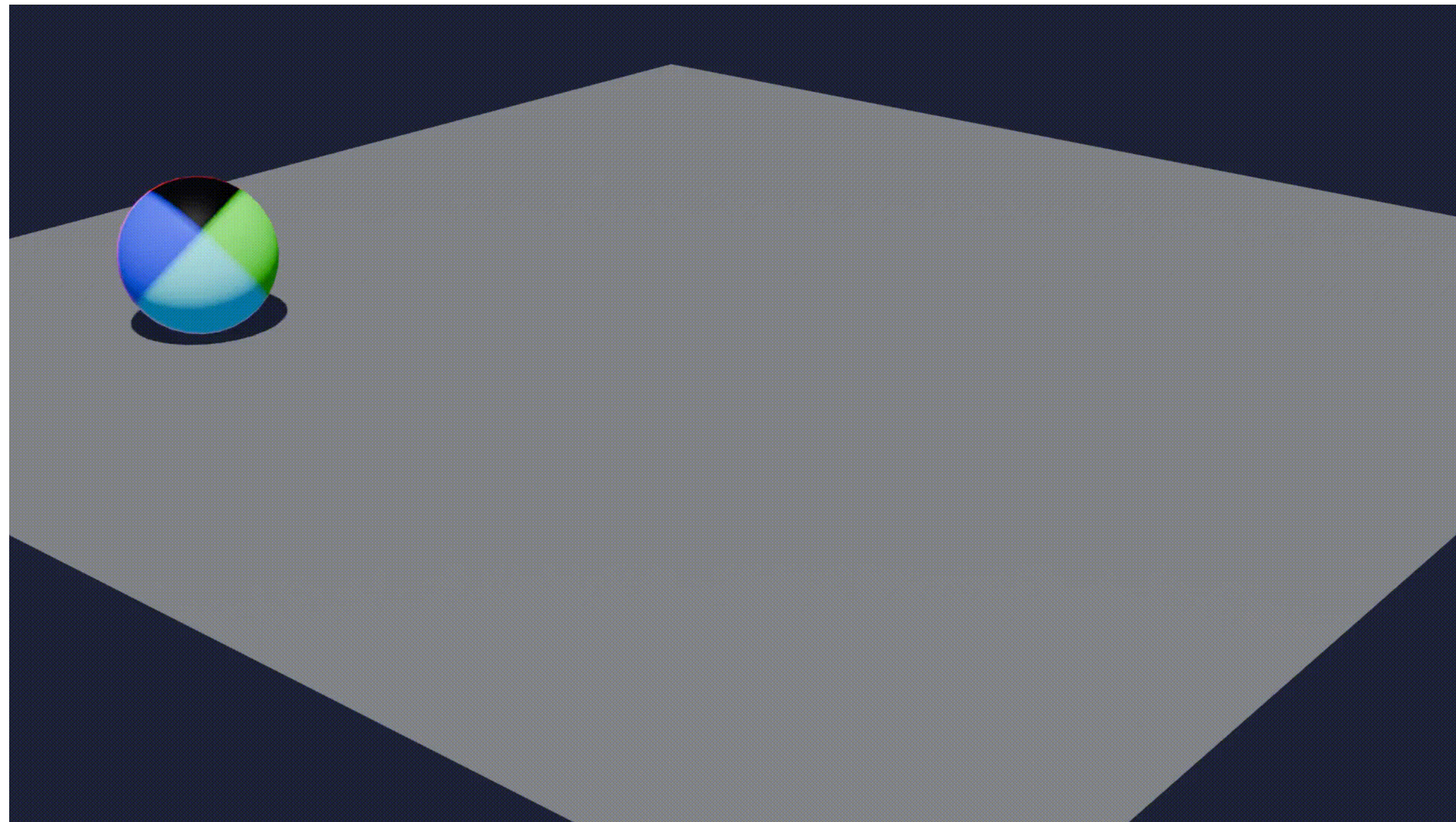
Holonomic / Integrable

Ball rolling without slipping simulated



Matching physical world

Using virtual forces
(Lagrange d'Alembert)



The optimal control point of view

Constrained optimal motion
(Vakonomic)

Simulations

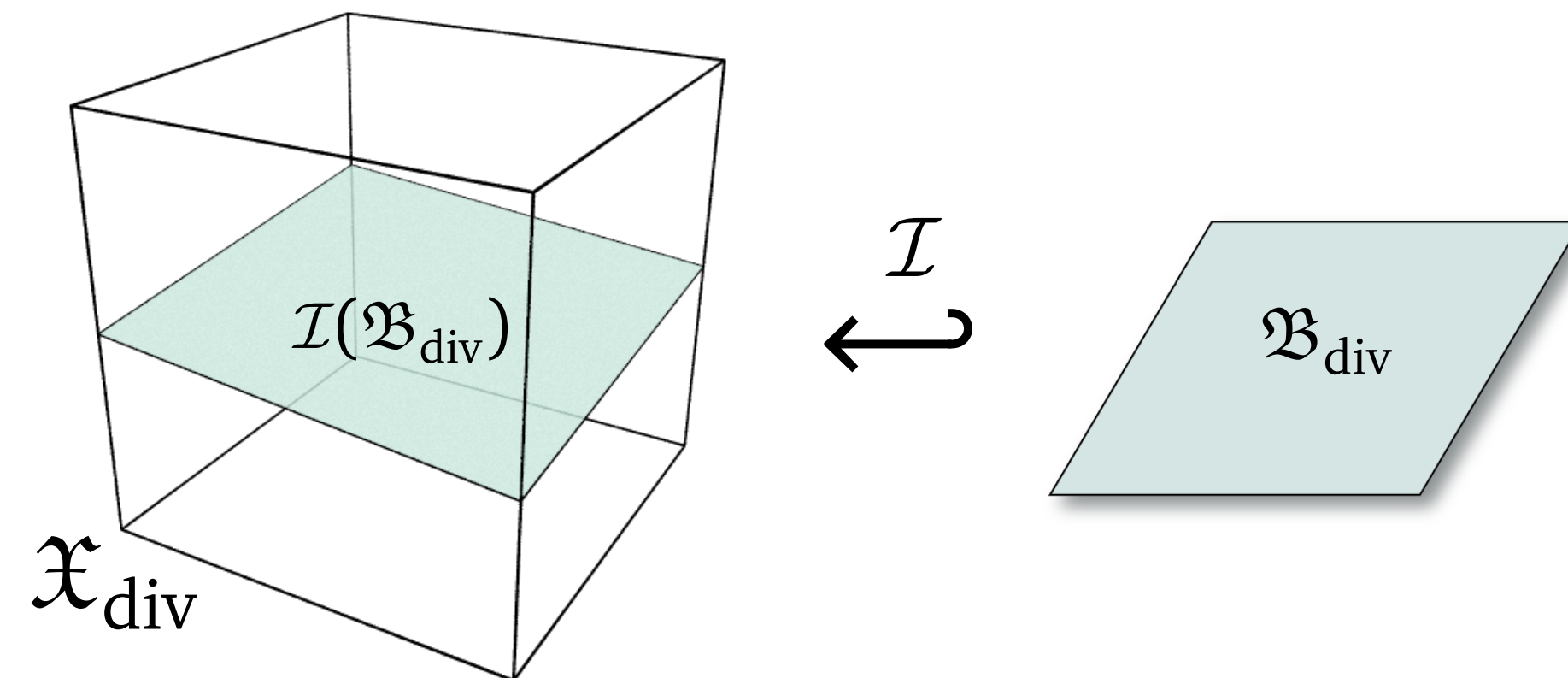
Physics with velocity constraints

$$\operatorname{argmin}_{x_t, u_t} \int_{[0,1]} \mathcal{L}(x_t, u_t) dt$$

$$\text{subject to } g(x_t) = 0$$

$$\text{subject to } h(u_t) = 0$$

- Discretized velocity field can represent only a subset of continuous velocities



Simulations

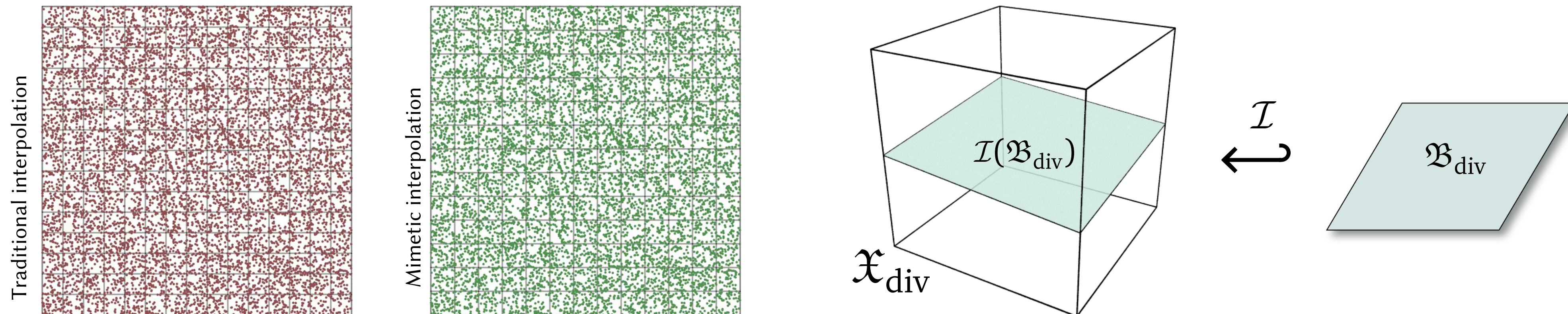
Physics with velocity constraints

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Simulations

Physics with velocity constraints

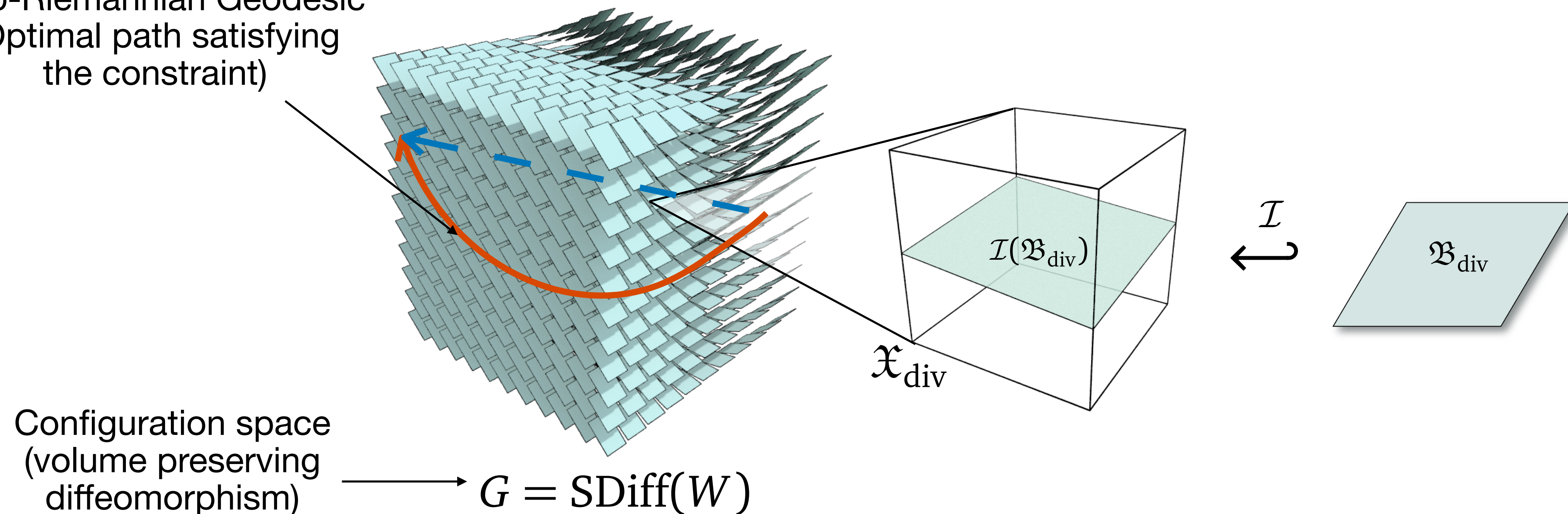
$$\operatorname{argmin}_{x_t, u_t} \int_{[0,1]} \mathcal{L}(x_t, u_t) dt$$

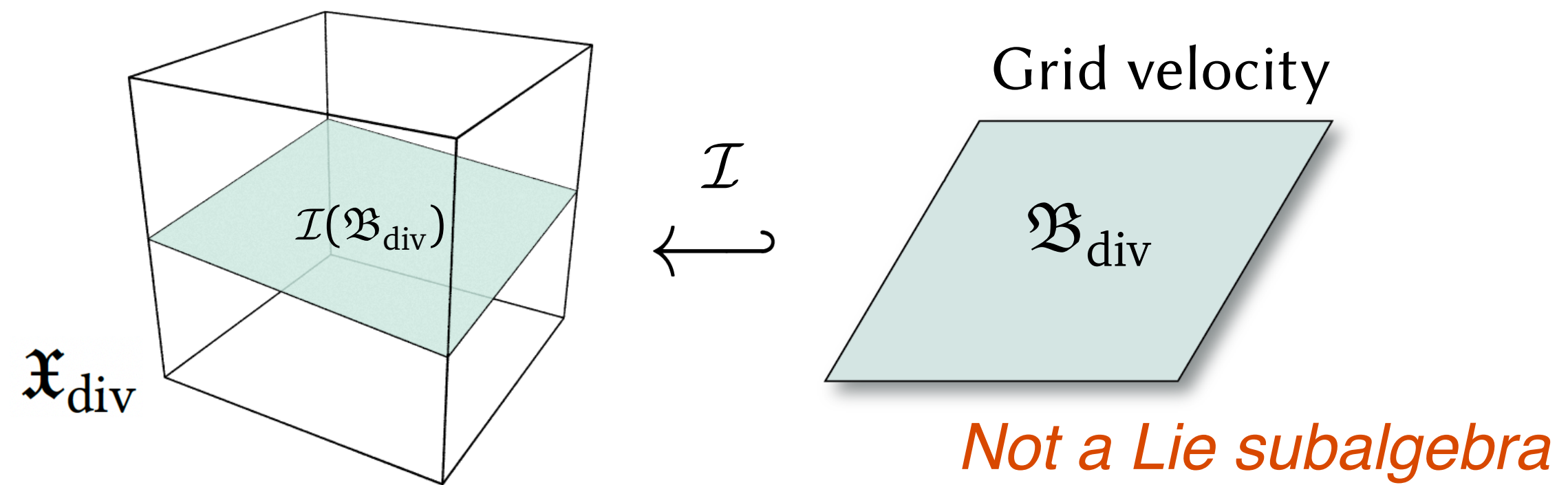
$$\text{subject to } g(x_t) = 0$$

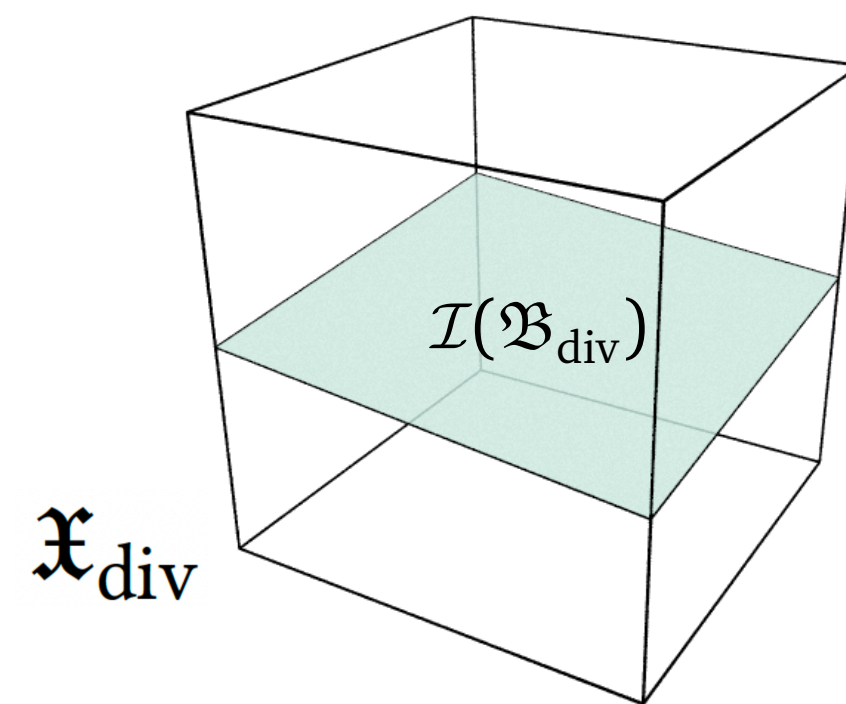
$$\text{subject to } h(u_t) = 0$$

- Discretized velocity field can represent only a subset of continuous velocities

Sub-Riemannian Geodesic
(Optimal path satisfying
the constraint)

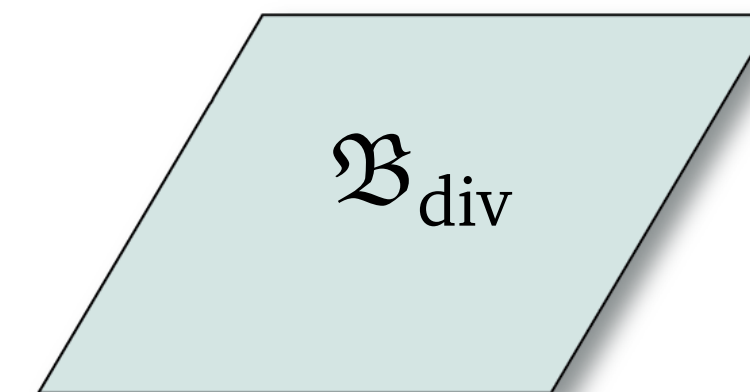






$\mathcal{I}$

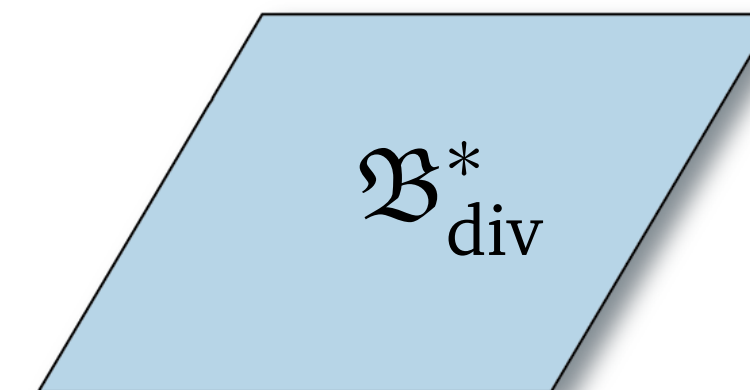
Grid velocity



Not a Lie subalgebra

⋮

Grid circulations

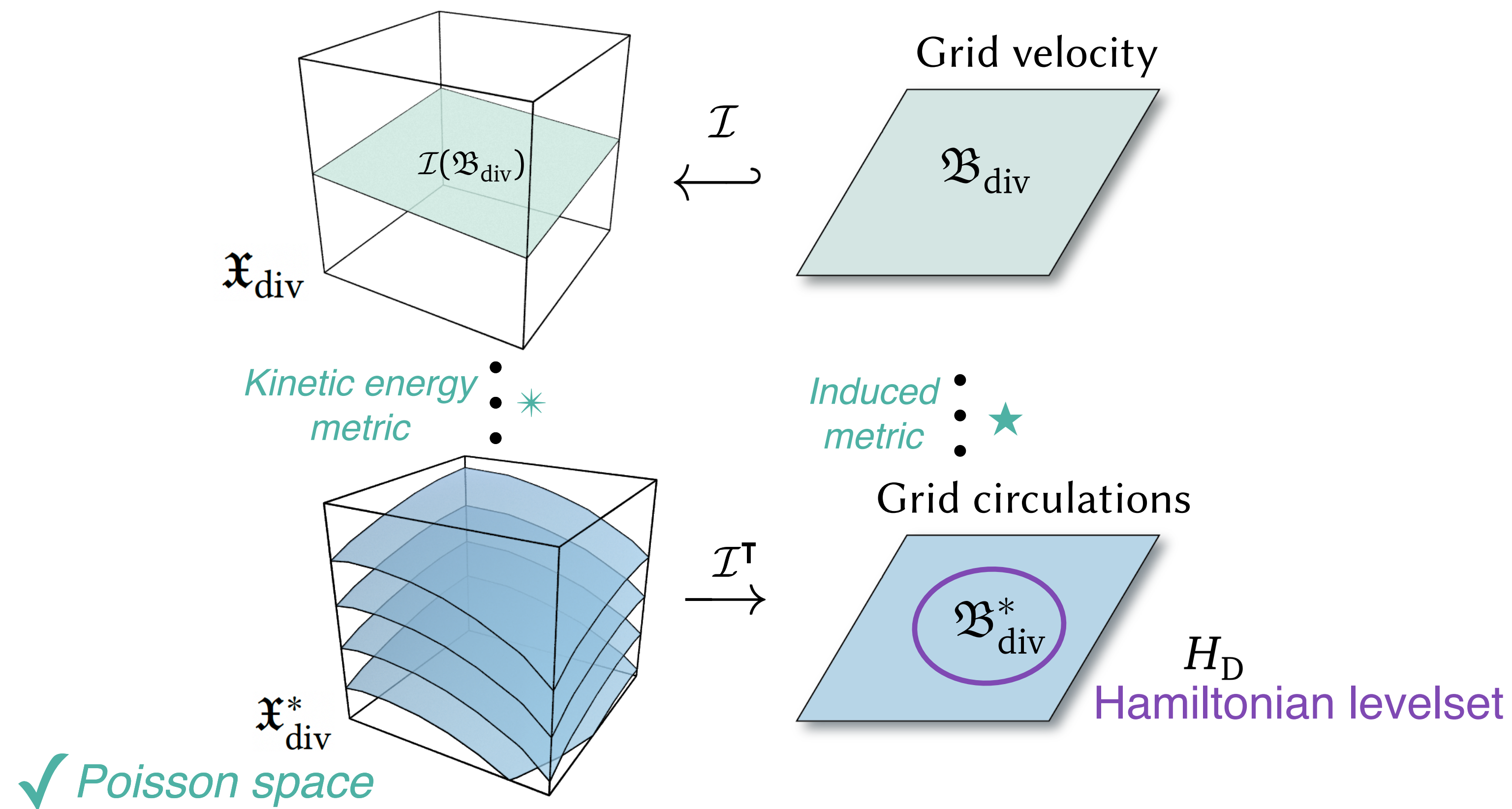


Not a Poisson space

No coadjoint orbits

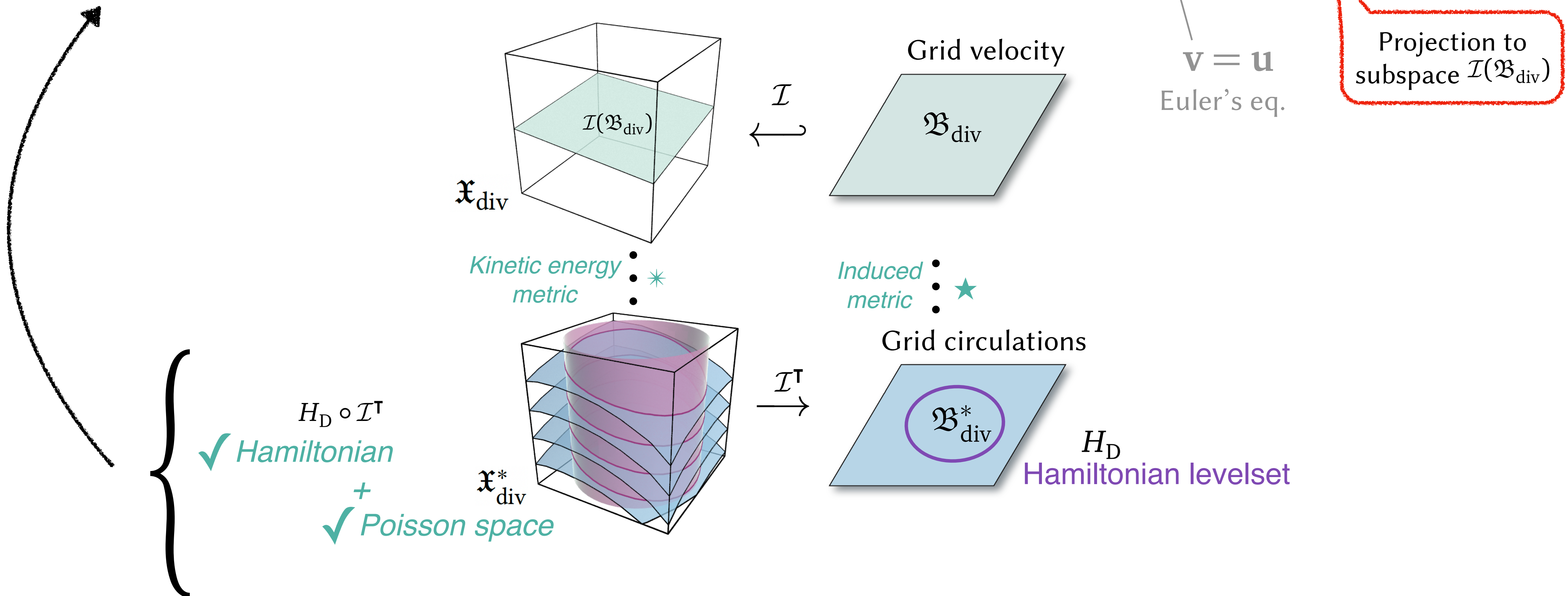
Kelvin's circulation theorem not respected

Vortical structures break



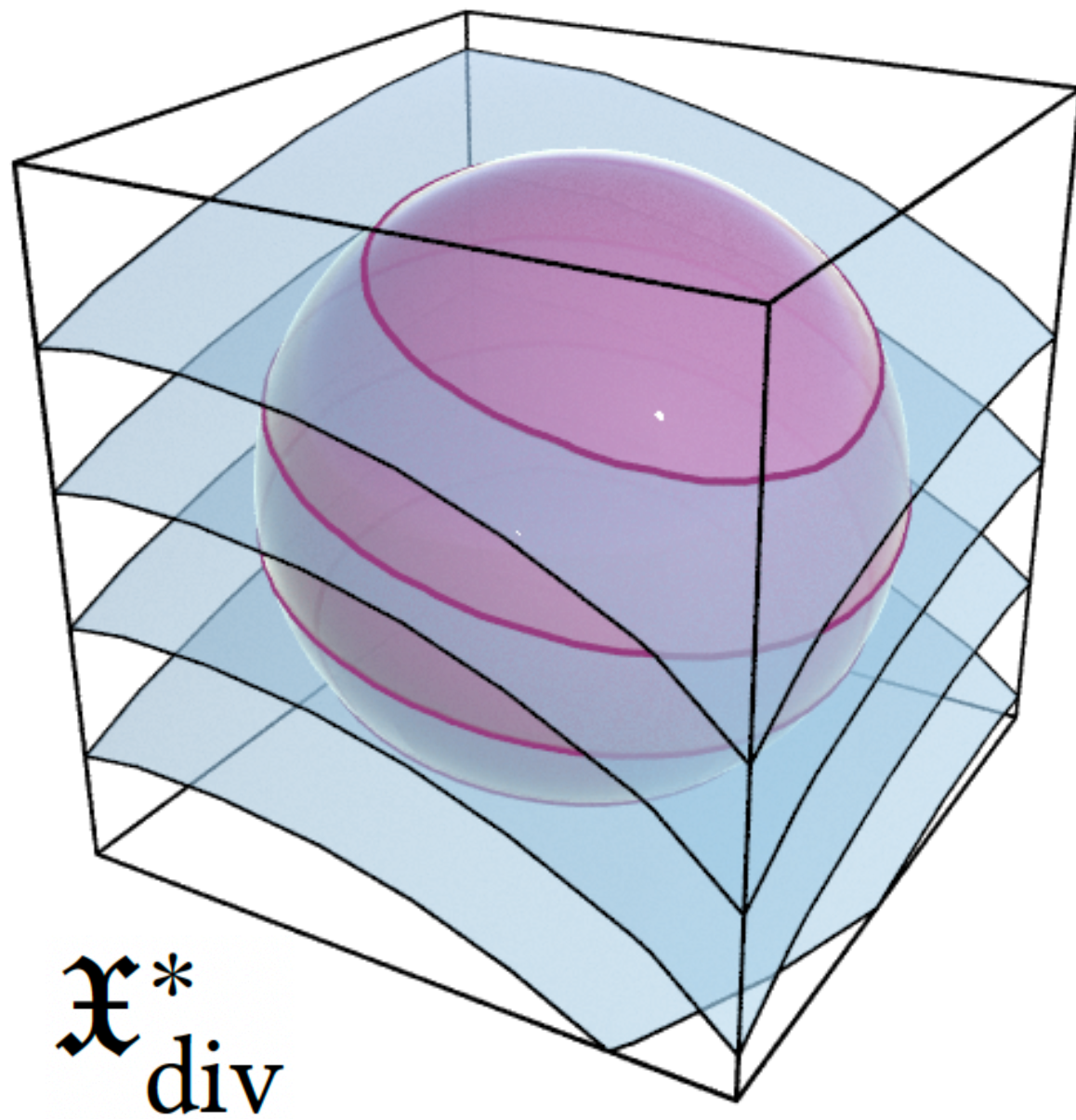
Definition ($\mathcal{I}$ -discrete Euler): The flow is given by the Hamiltonian system $(\mathfrak{X}_{\text{div}}^*, H_D \circ \mathcal{I}^\top)$,

and its equation of motion is given by:
$$\begin{cases} \frac{\partial \mathbf{u}^b}{\partial t} + \mathcal{L}_{\mathbf{v}} \mathbf{u}^b = -dq \\ \nabla \cdot \mathbf{u} = 0 \end{cases}, \text{ where } \mathbf{v} = \mathcal{I} P_{\mathfrak{B}_{\text{div}}} \star^{-1} \mathcal{I}^\top \mathbf{u}^b$$

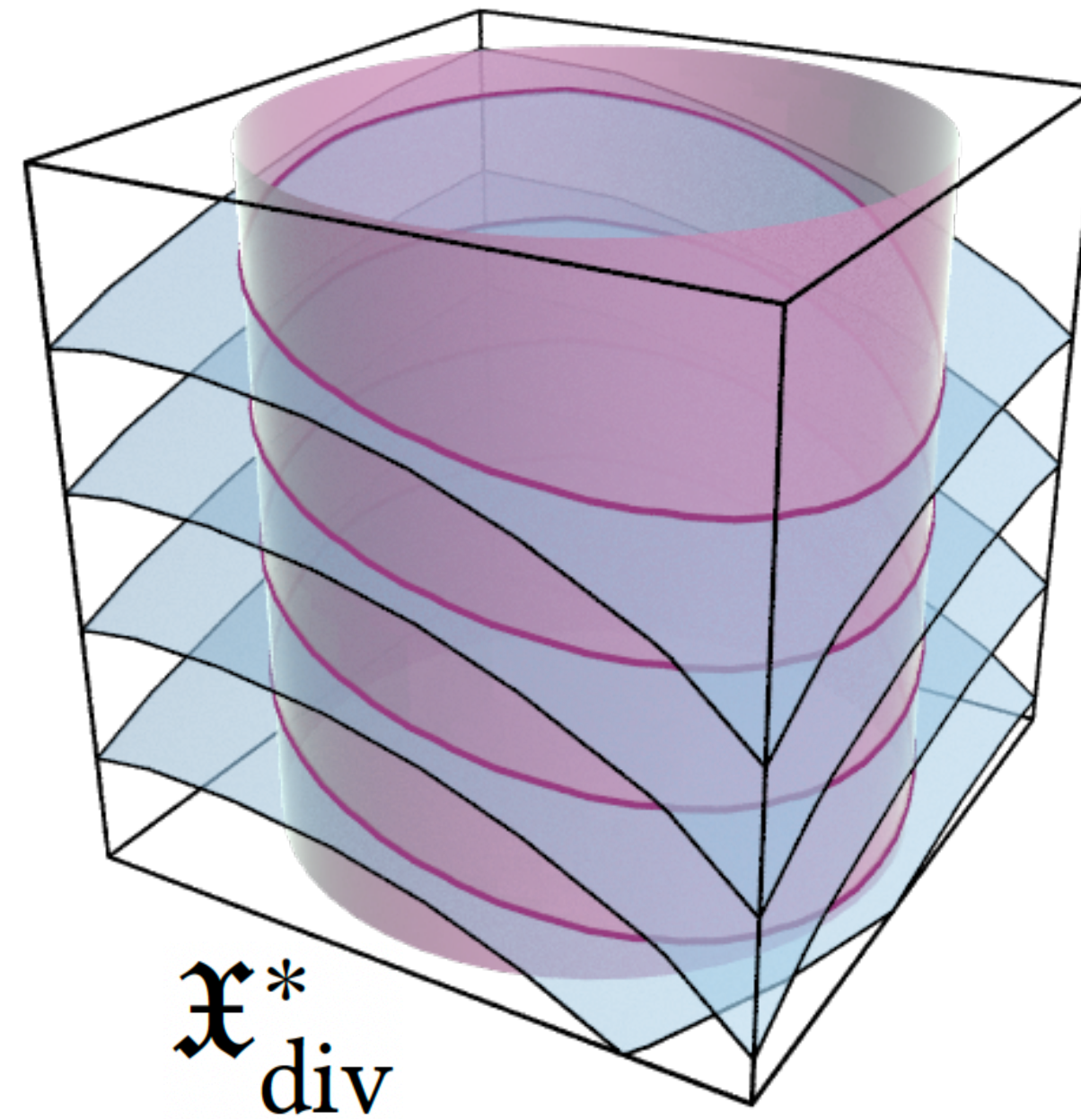


Ours vs the continuous formulation

Original Euler formulation



Our $\mathcal{I}$ -discrete Euler formulation



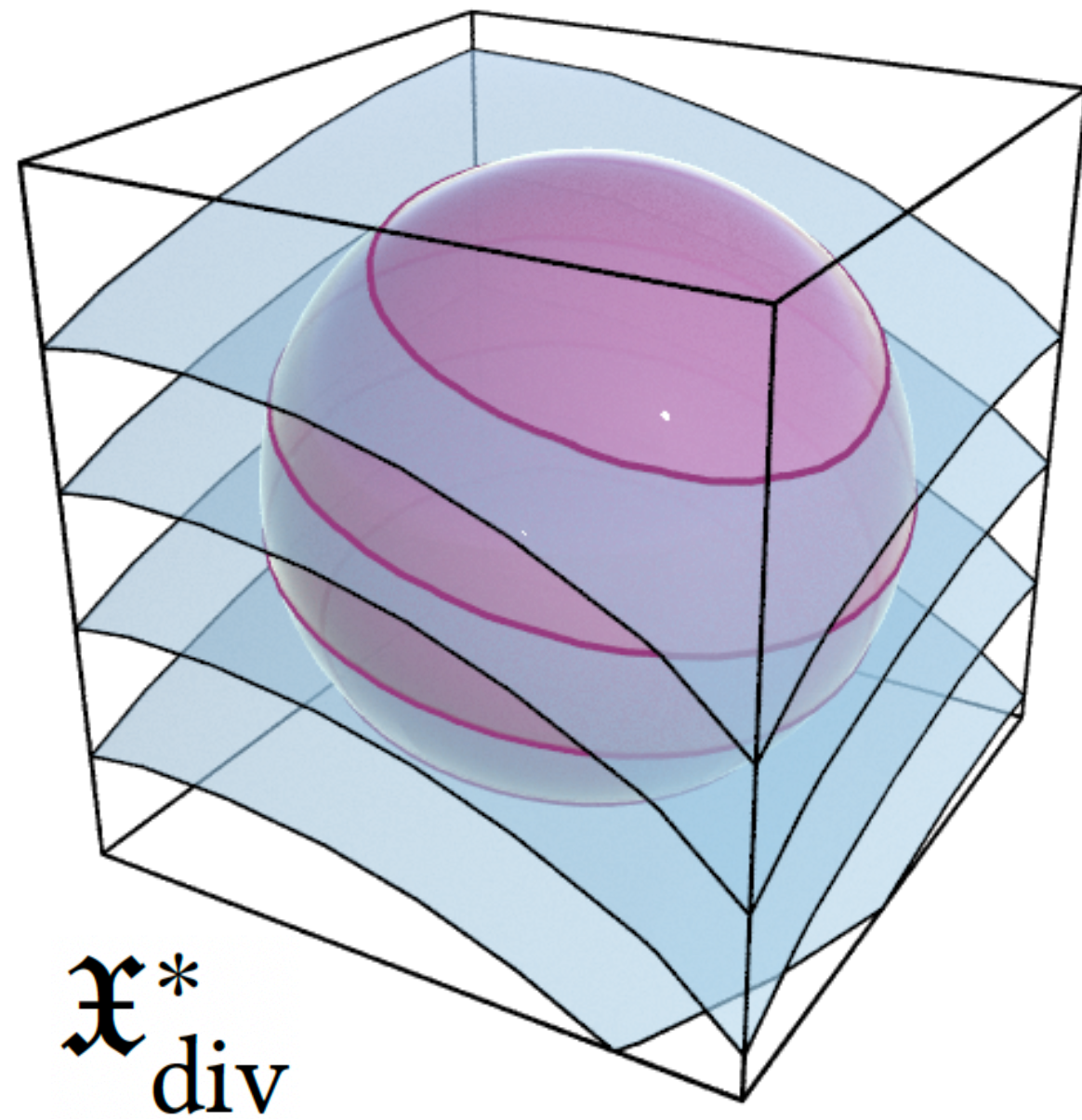
Energy level sets

Coadjoint orbits

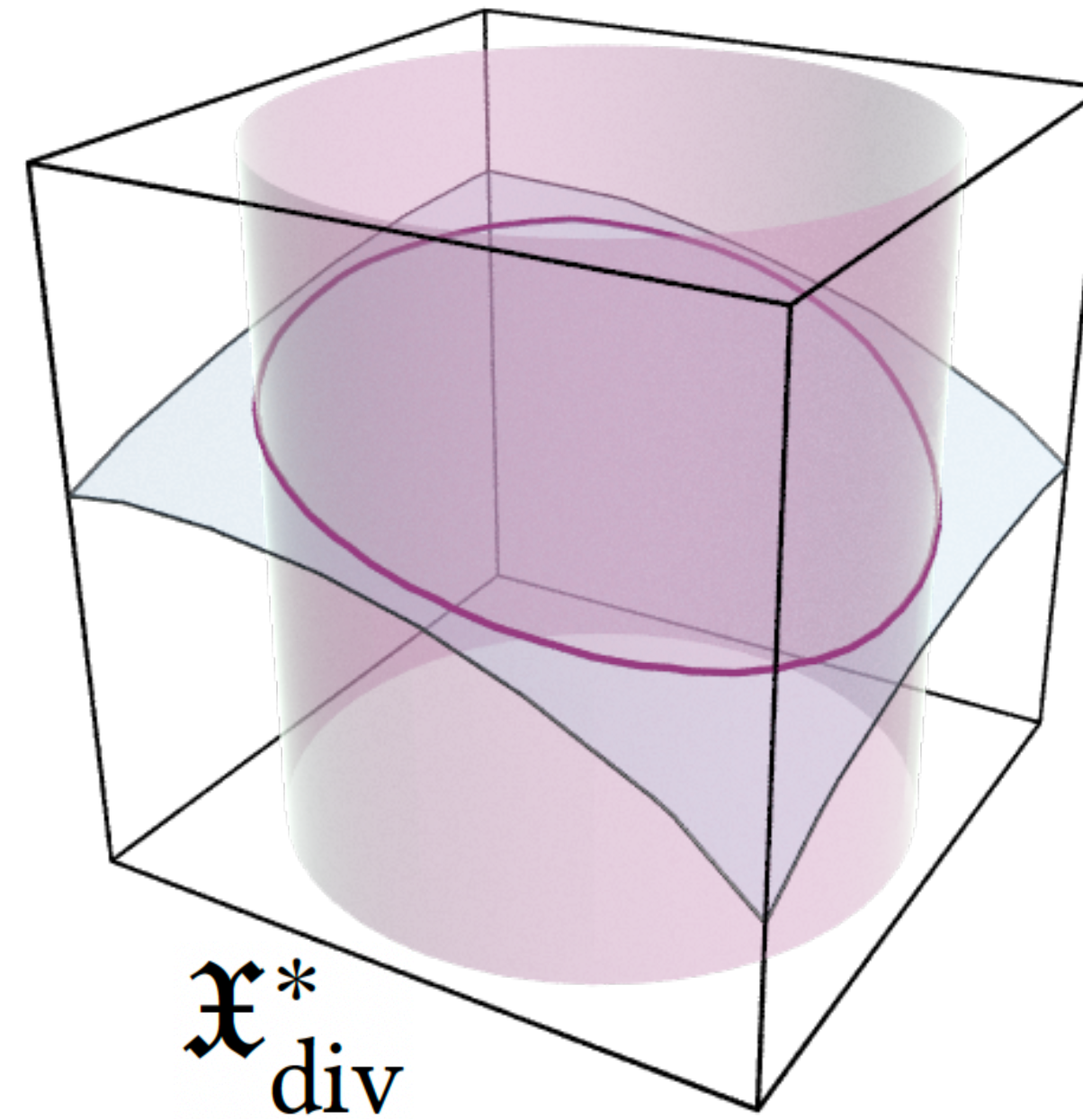
Theorem: The $\mathcal{I}$ -discrete Euler flow preserves the energy and coadjoint orbits in $\mathfrak{X}_{\text{div}}^*$.

Ours vs Previous methods

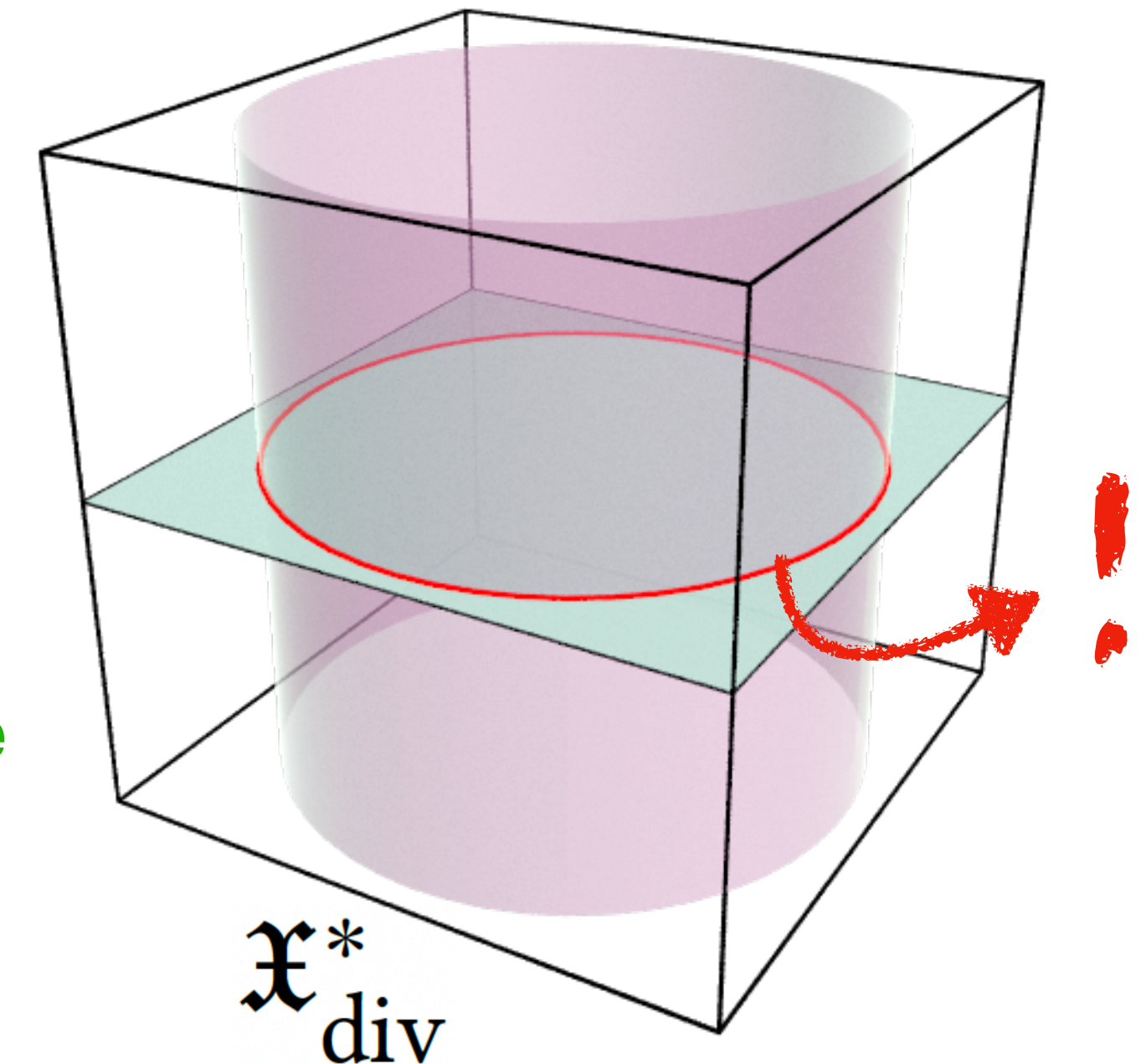
Original Euler formulation



Our I-discrete Euler formulation



Previous FEM-based formulations



Energy level sets

Coadjoint orbits

Discretization
constraint subspace

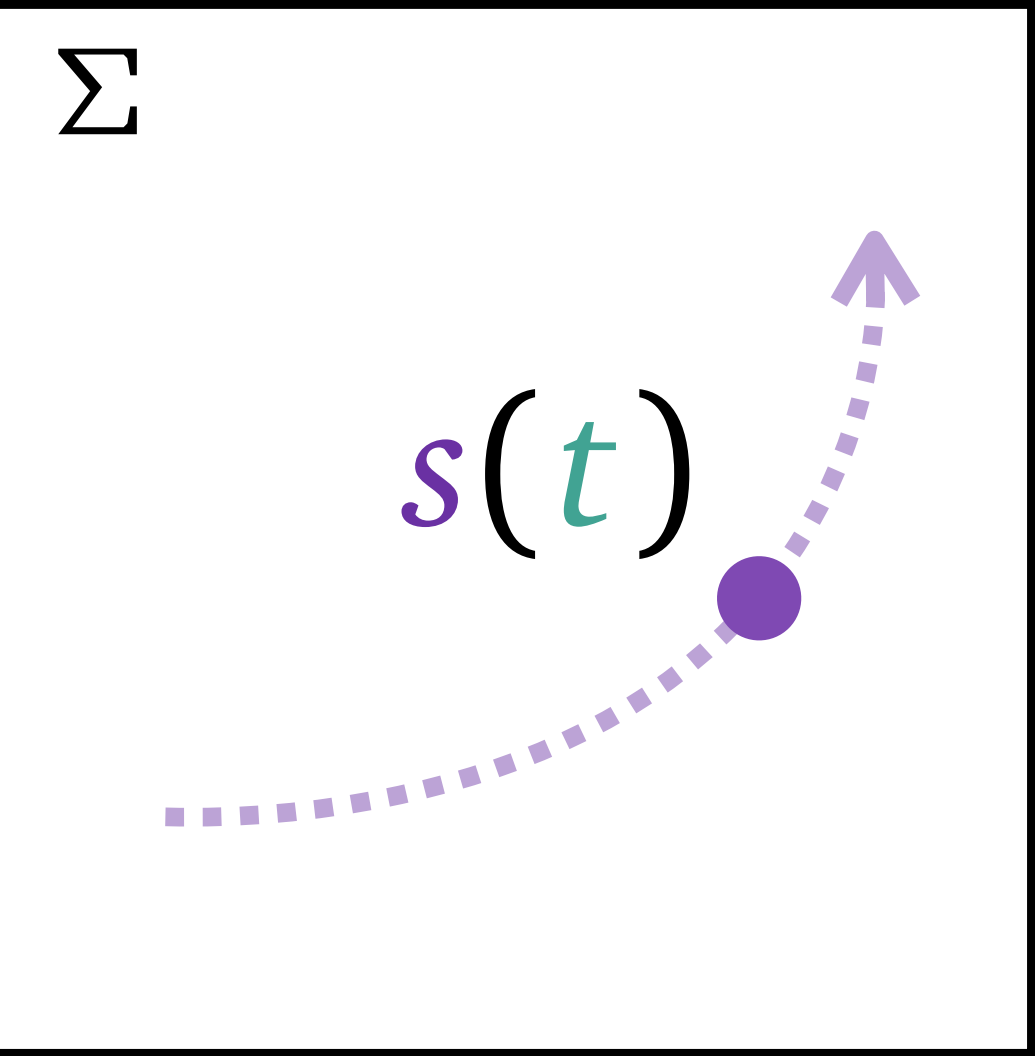
- ✓ Constrained geodesics (Sub-Riemannian)
- ✓ Constrained Variational (Vakonomic)
- ✓ Coadjoint preserving

- ✗ Not geodesics
- ✗ Not variational (Lagrange-d'Alembert)
- ✗ Not coadjoint preserving

Non-holonomic
systems of Variational
Axiomatic Kind

Question: How to formulate a structure-preserving numerical scheme for the $\mathcal{I}$ -discrete Euler's equation?

Auxiliary Symplectic space



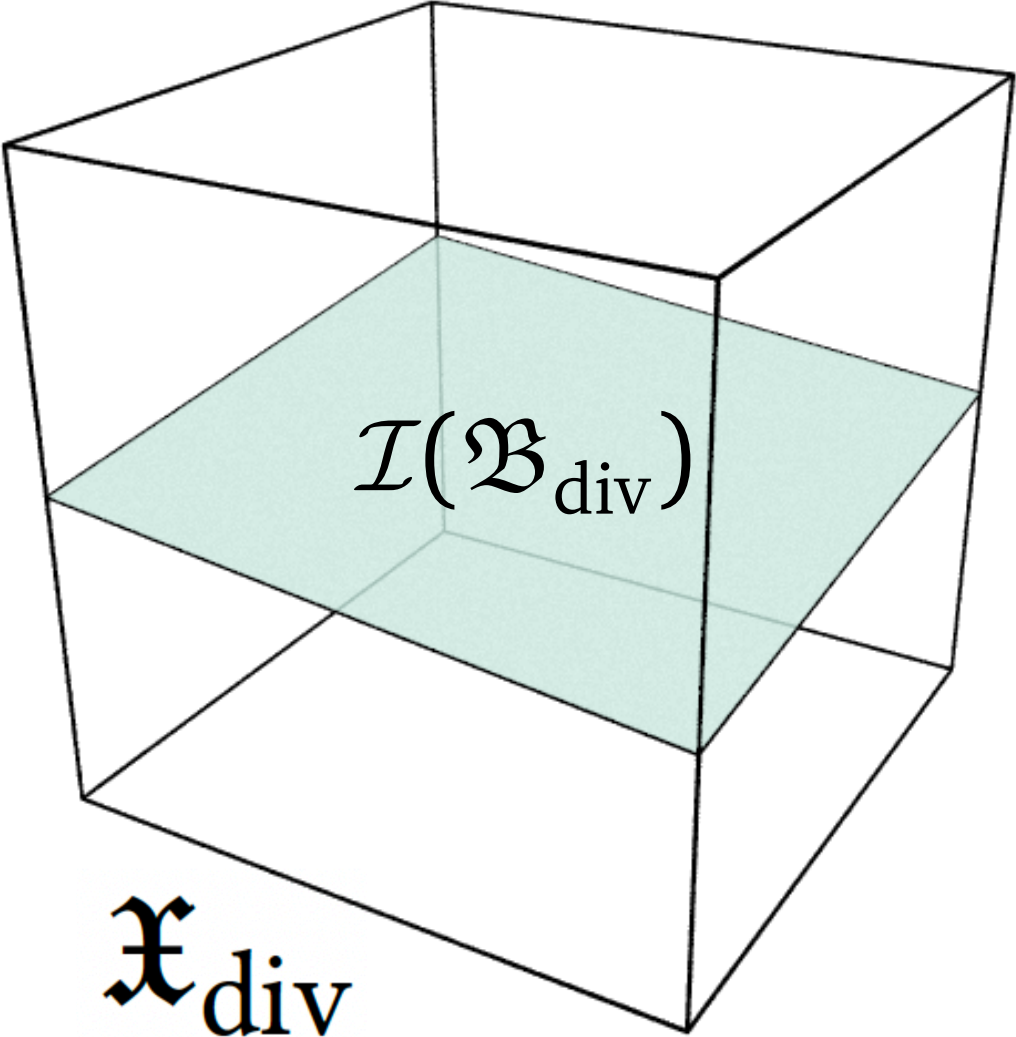
Numerically tractable space

Equivalent dynamics between Σ and $\mathfrak{X}_{\text{div}}^*$

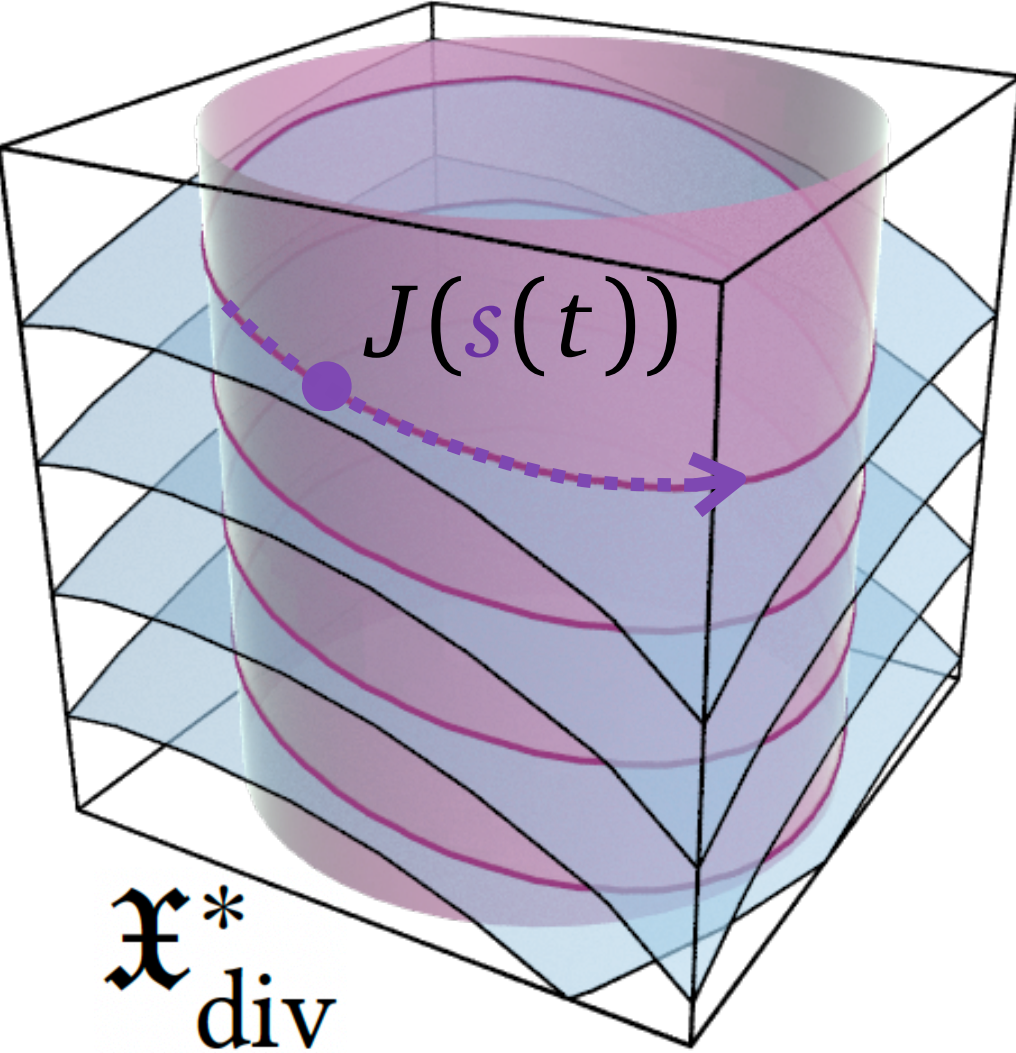
Symplectic advection Φ_*

Momentum map J

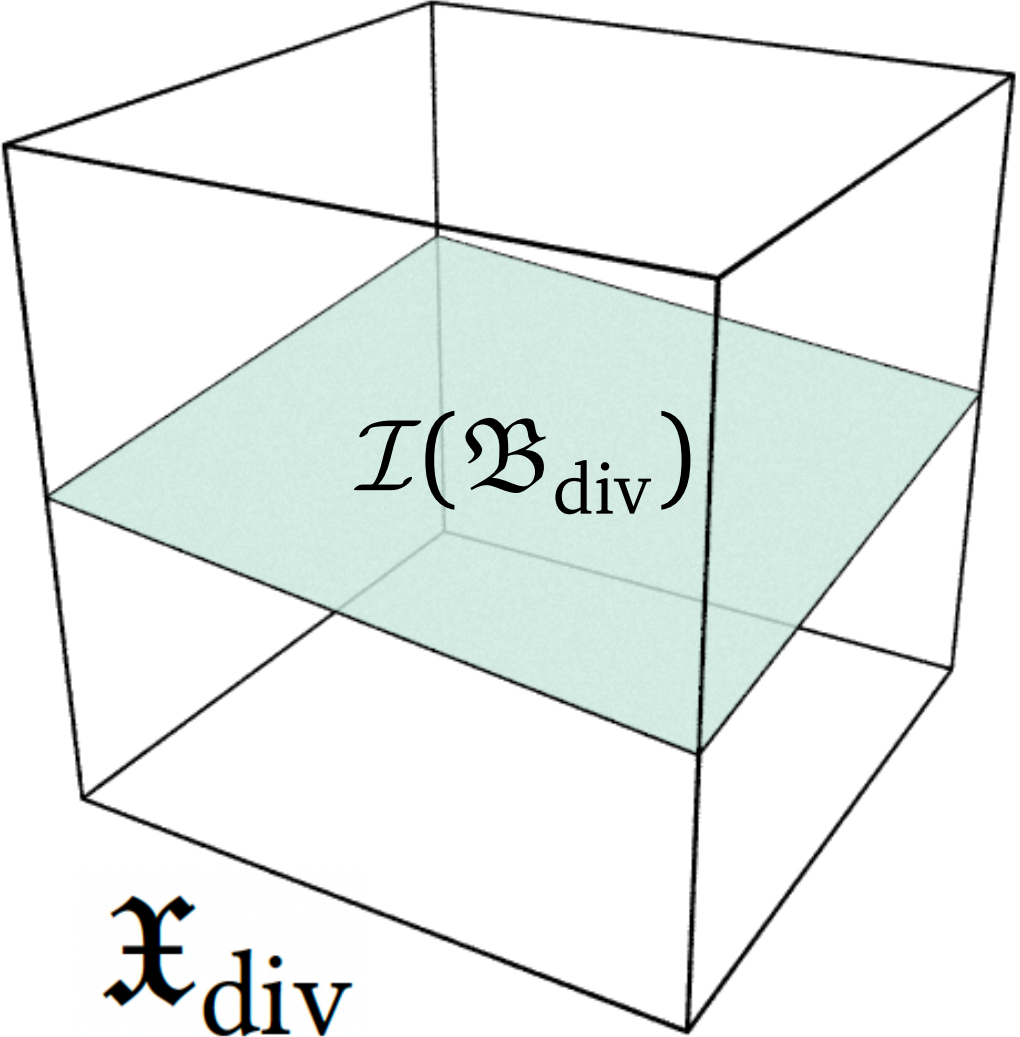
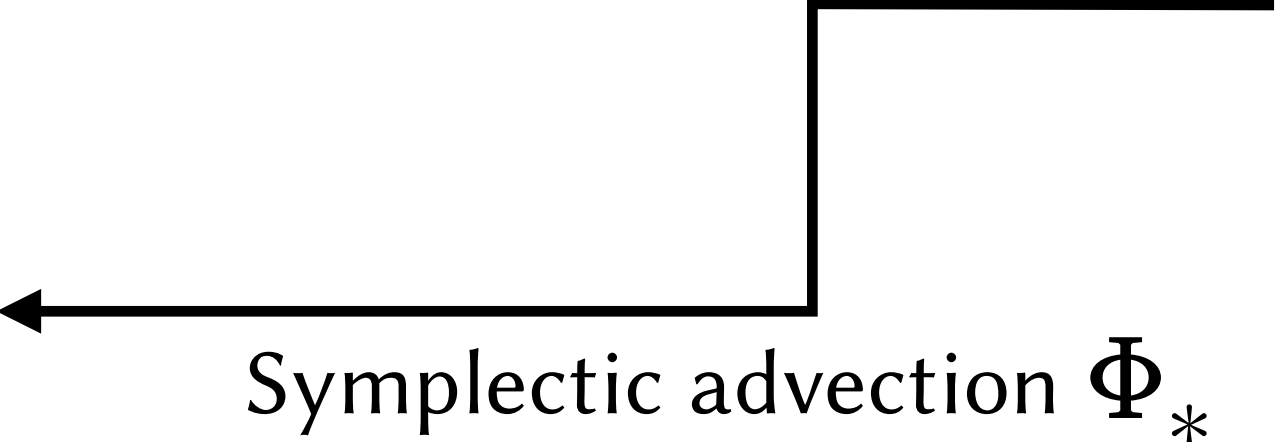
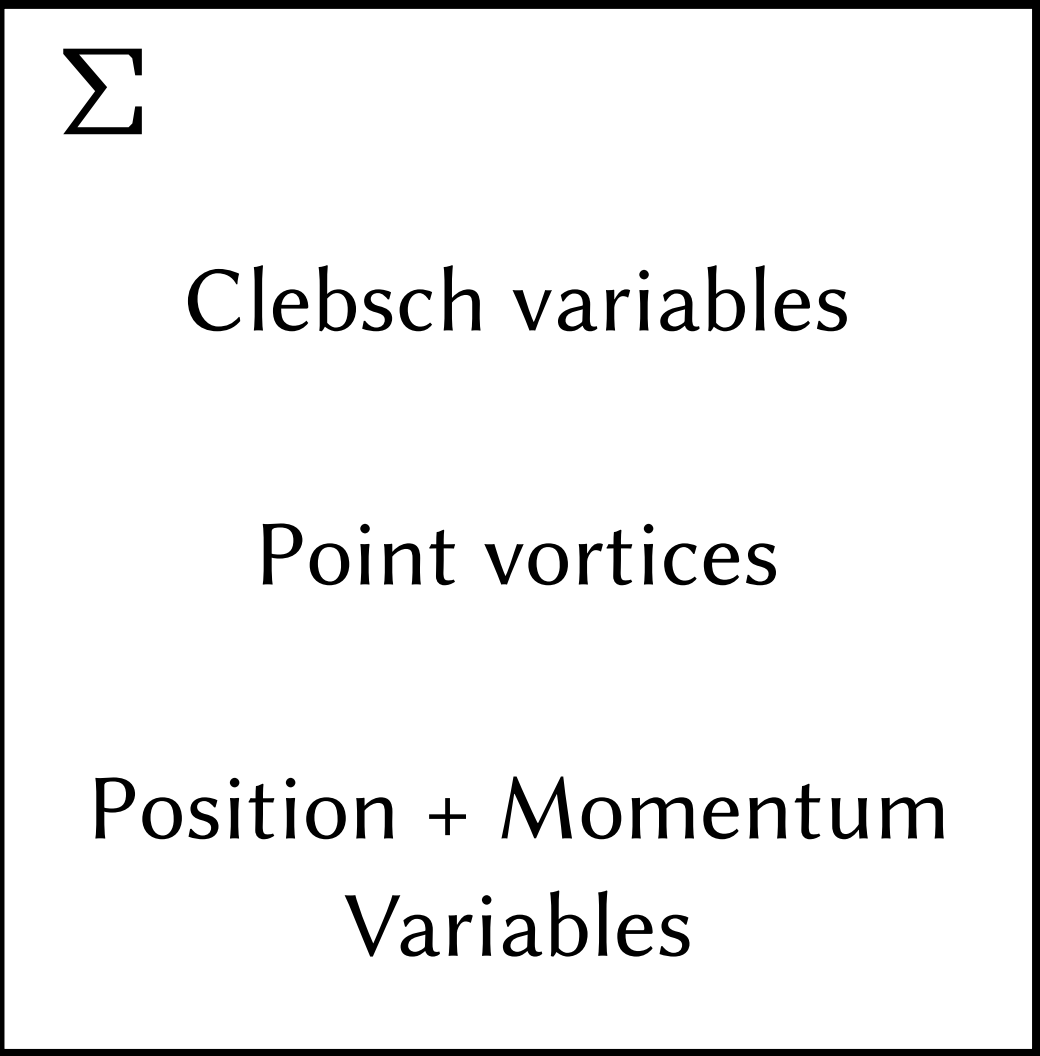
Canonically defined



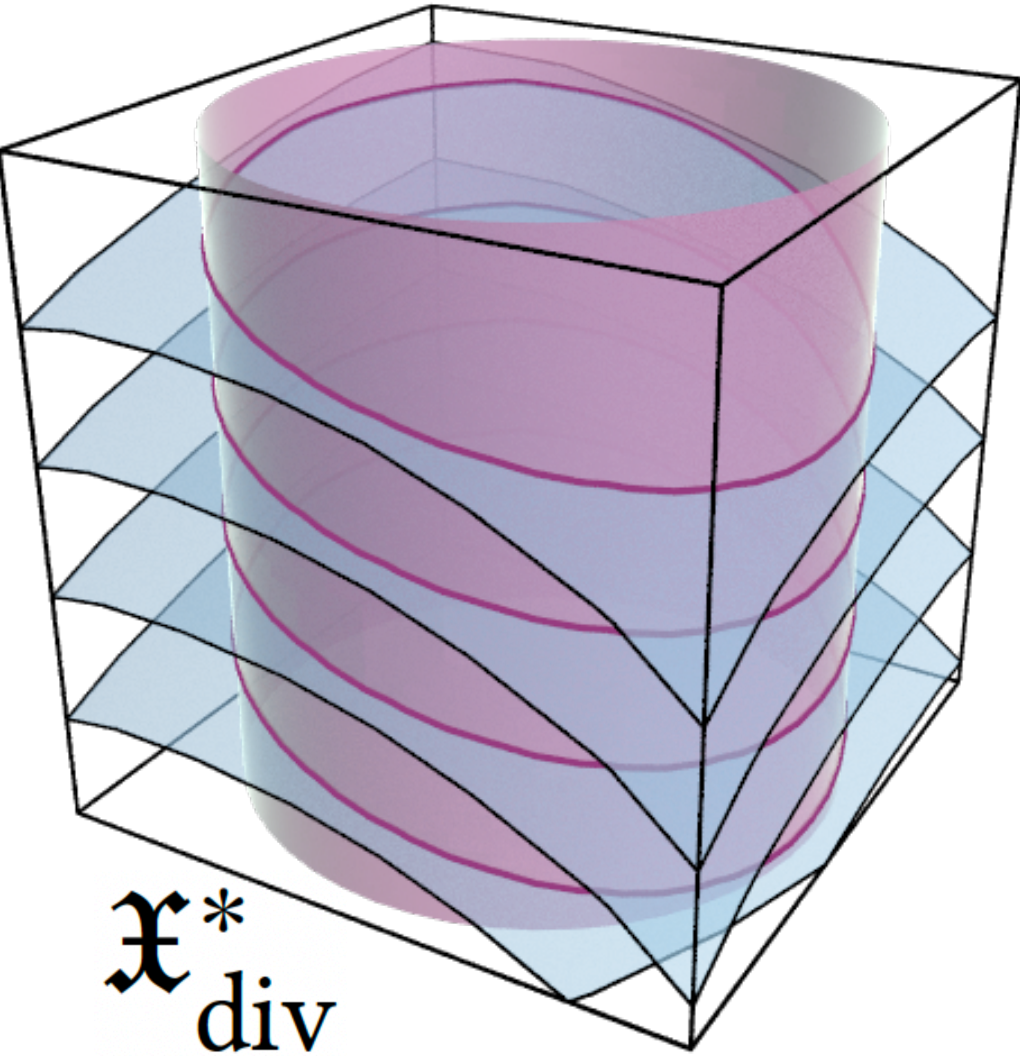
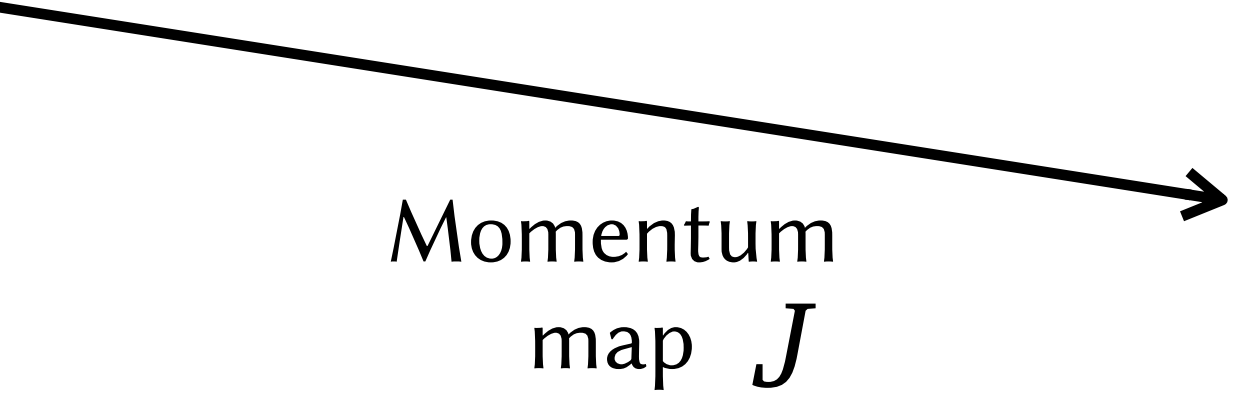
⋮



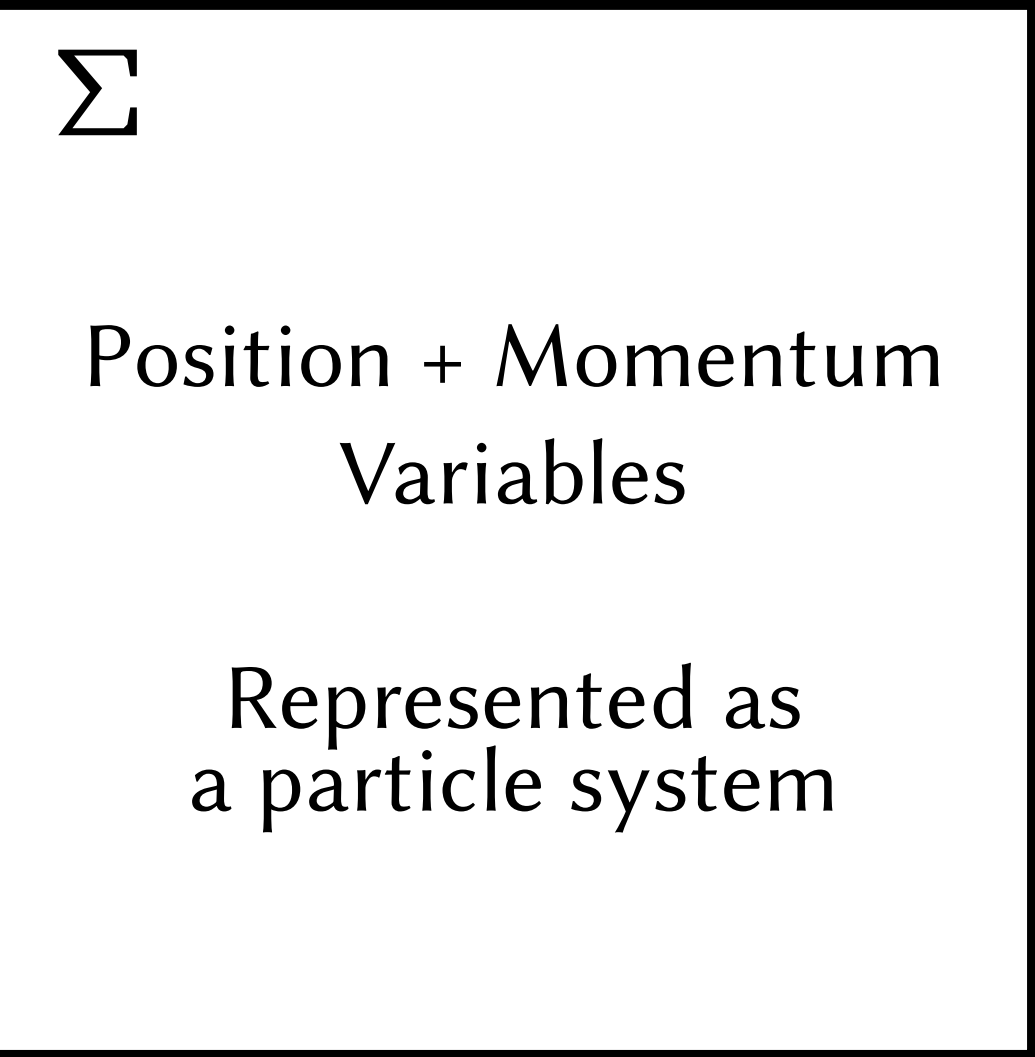
Auxiliary Symplectic space



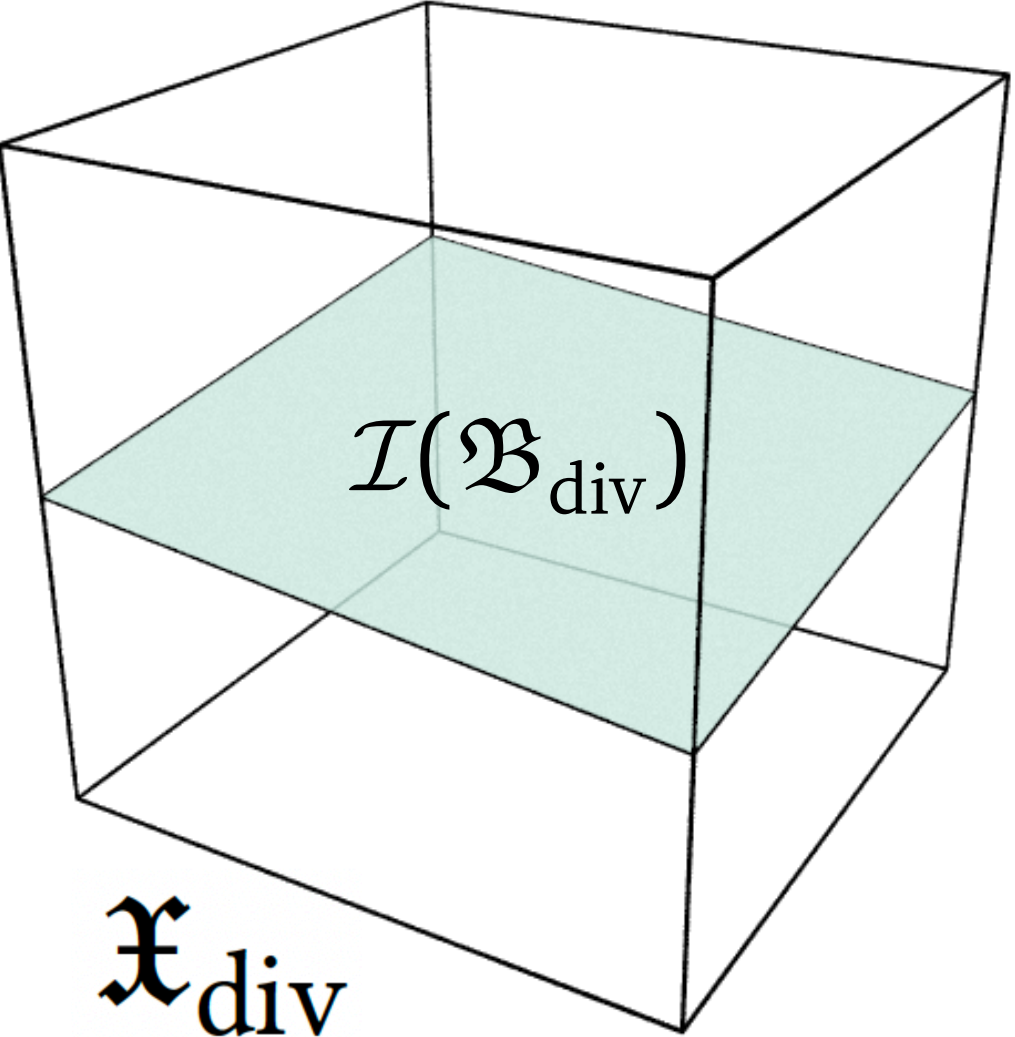
⋮



Auxiliary Symplectic space



Symplectic advection Φ_*

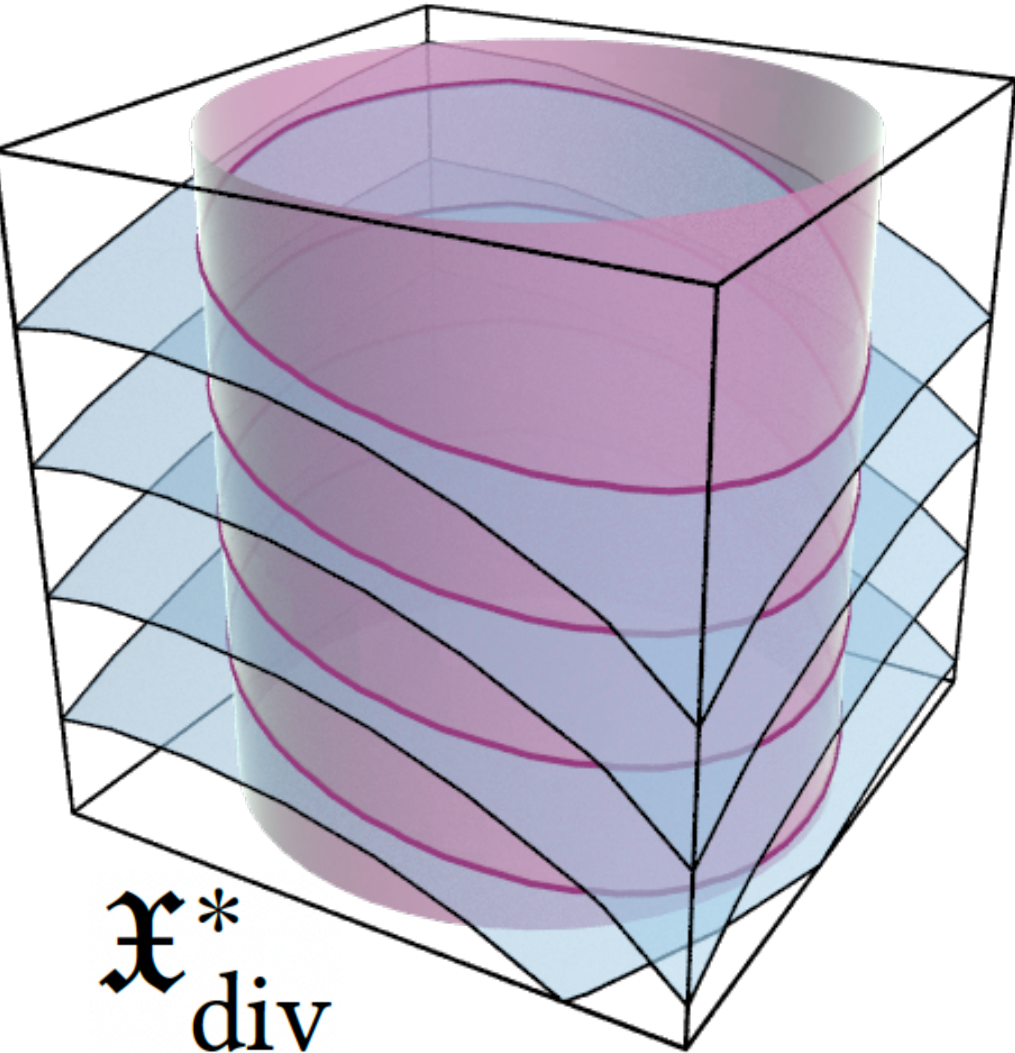


$\mathfrak{X}_{\text{div}}$

$\vdots$

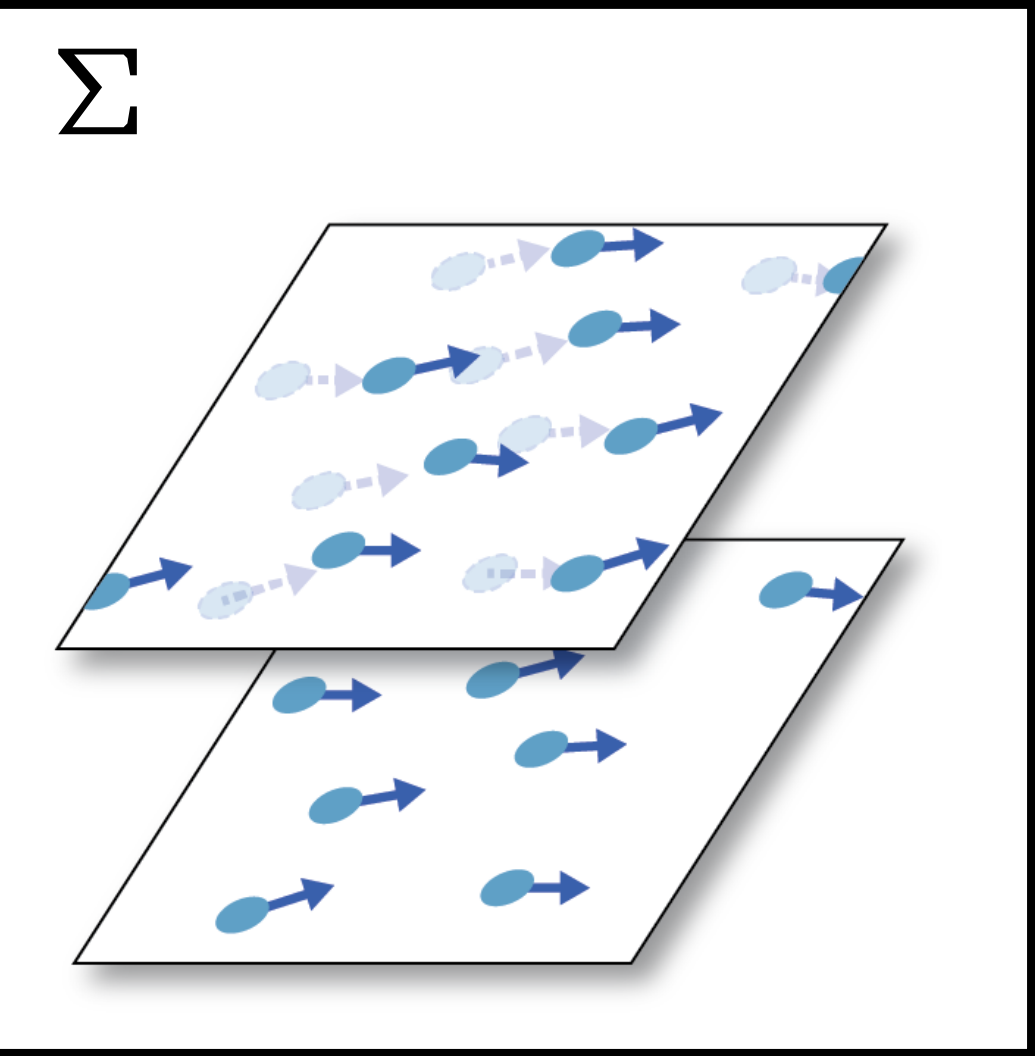


Momentum
map J



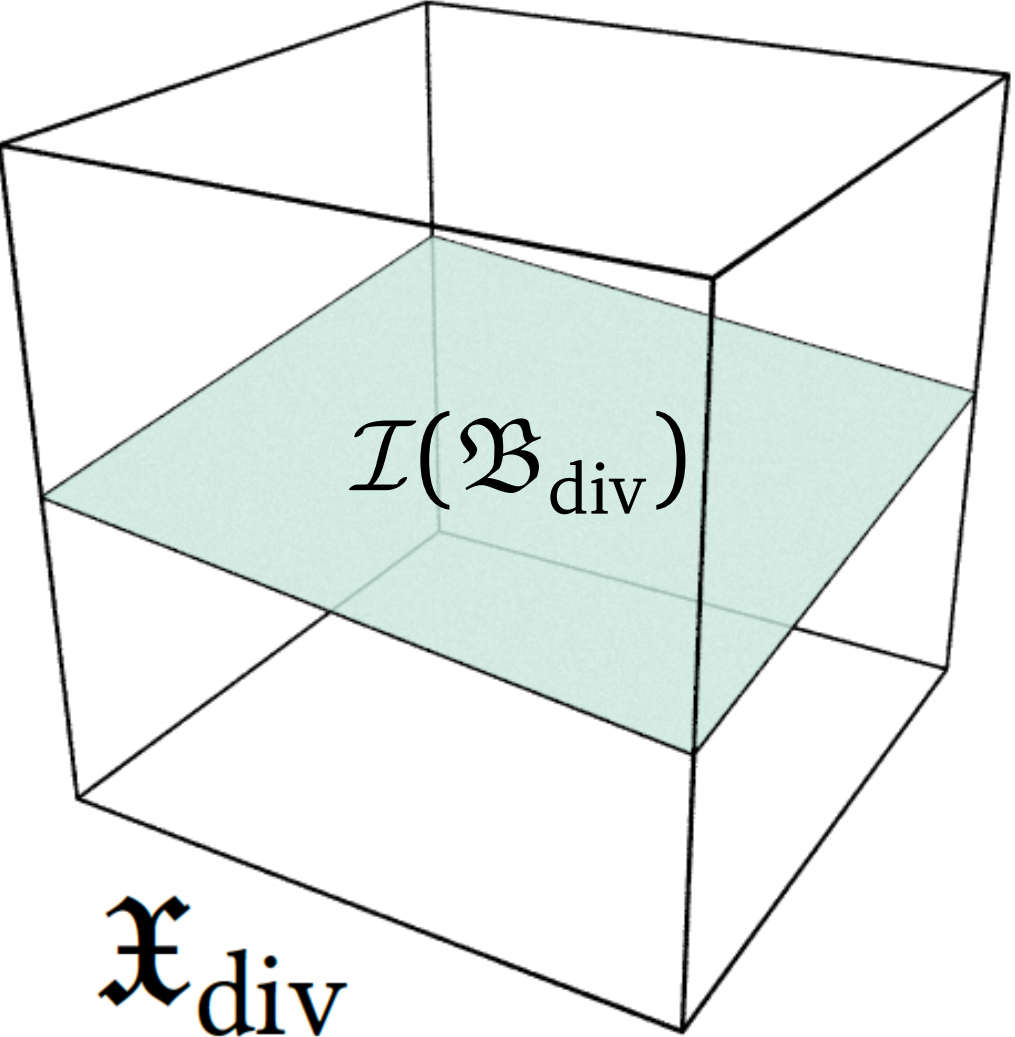
$\mathfrak{X}_{\text{div}}^*$

Auxiliary Symplectic space

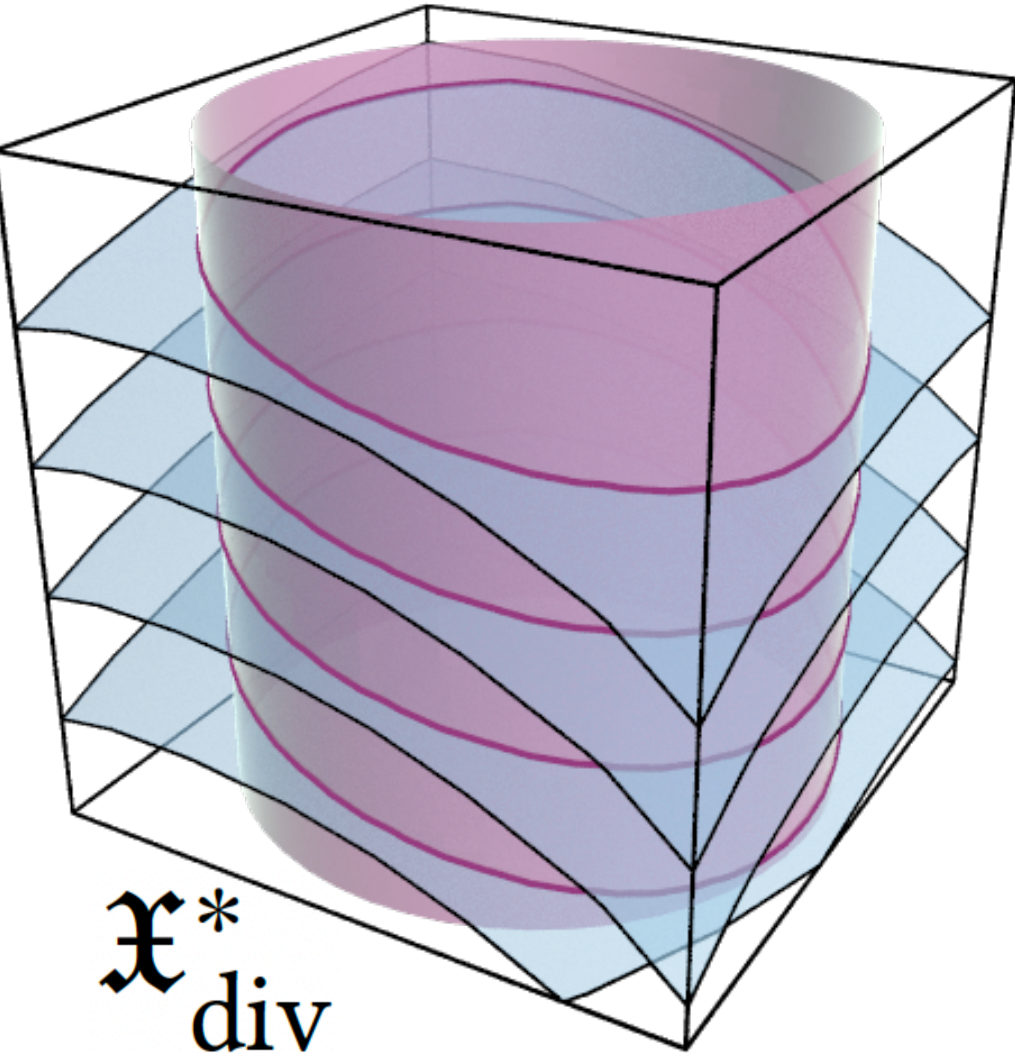


Symplectic advection Φ_*

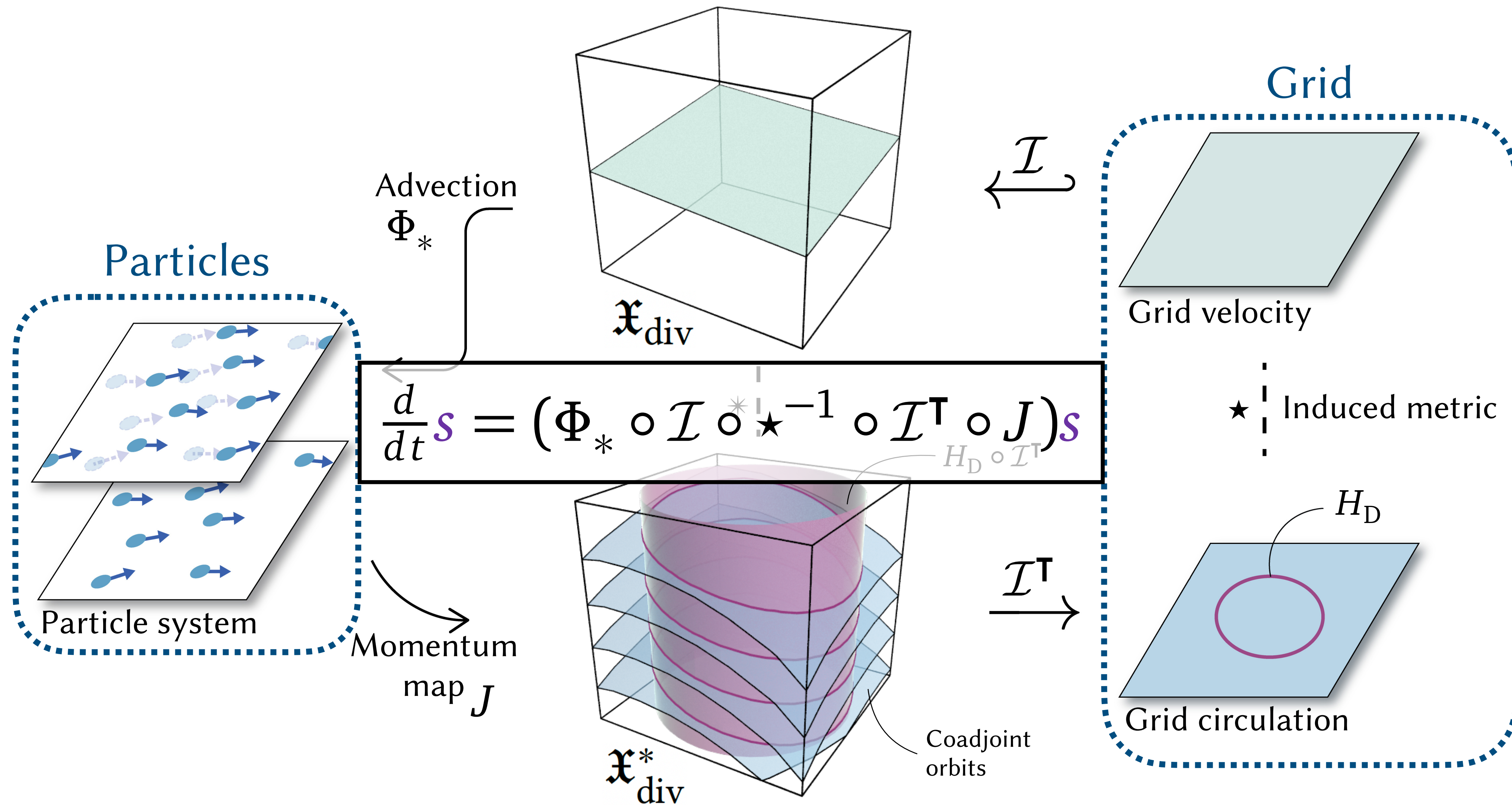
Momentum
map J



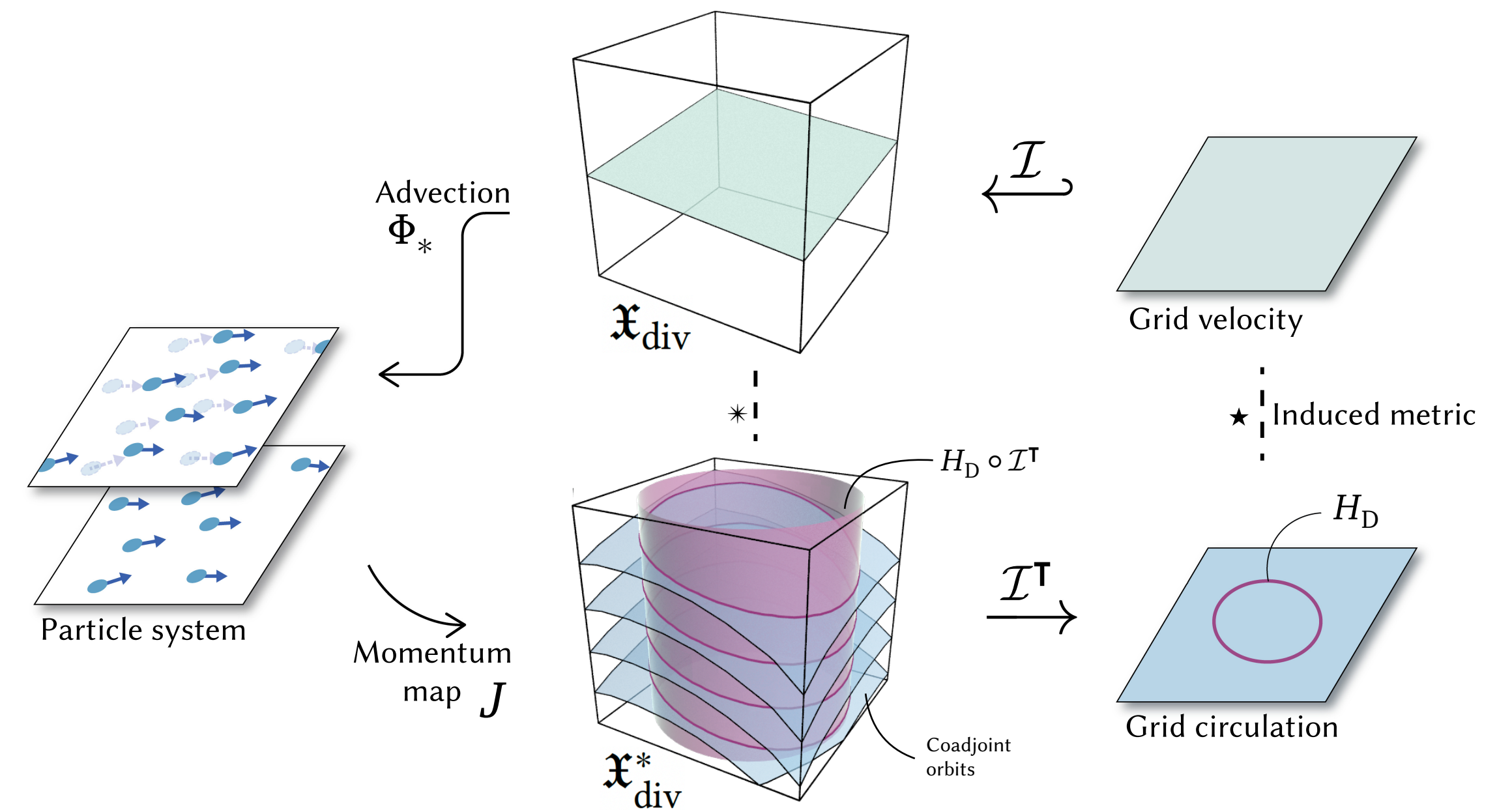
⋮



We call our method: Coadjoint Orbit Fluid Implicit Particle (CO-FLIP).



- We use **one single high-order divergence-free interpolation** $\mathcal{I}$; the rest are derived from it.
- Advection is **Lie group integrator**.
- Particle-to-grid transfer is **canonically** derived.
- Pressure projection respects the **induced metric**.
- This results in a Hamiltonian ODE that we **geometrically** integrate.
- The CO-FLIP algorithm preserves **coadjoint orbits** and **energy**.



$$\frac{d}{dt} \mathbf{s} = (\Phi_* \circ \mathcal{I} \circ \star^{-1} \circ \mathcal{I}^\top \circ J) \mathbf{s}$$

Results

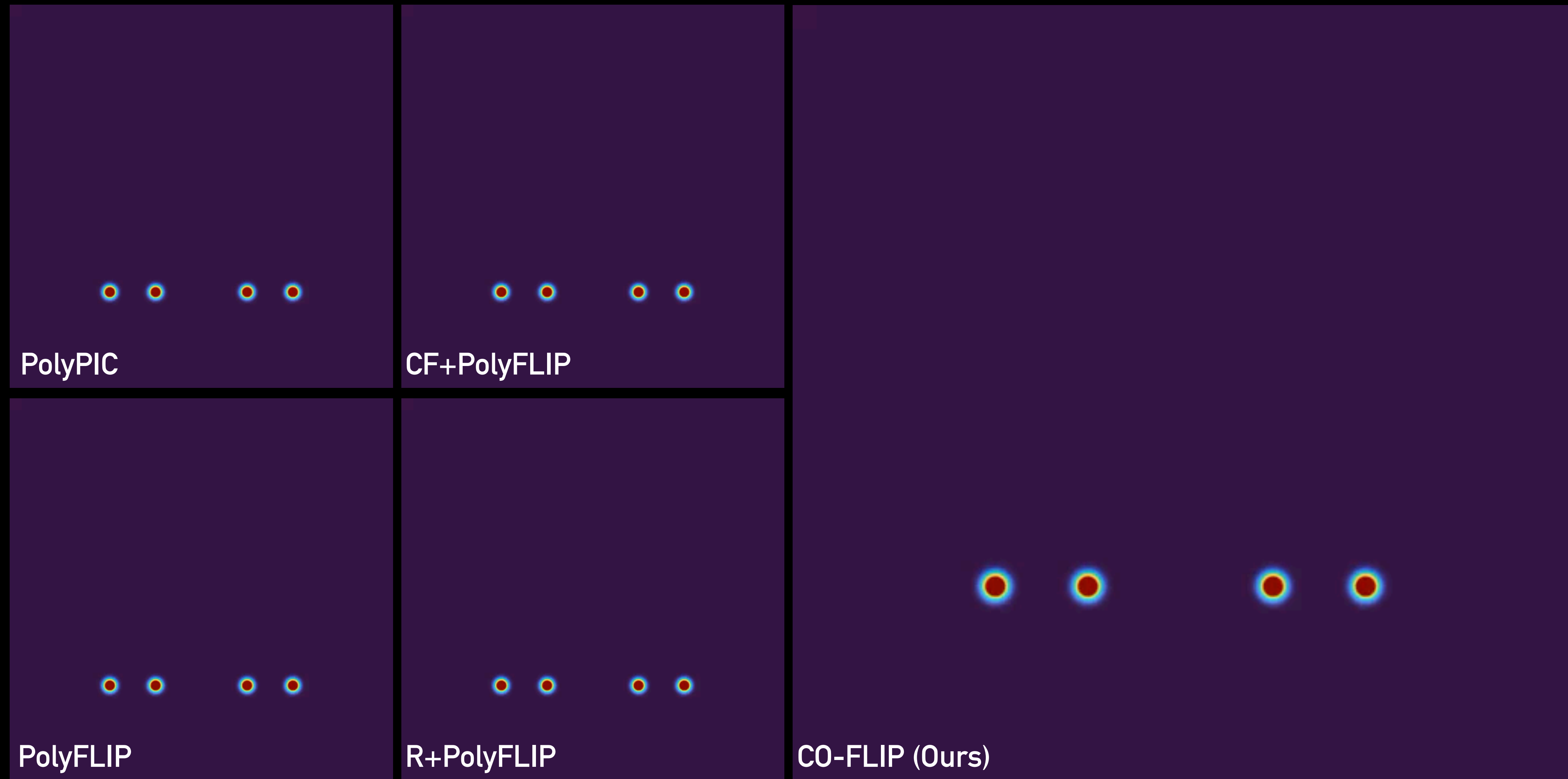
Pyroclastic cloud
simulated with a
previous method
at res 128x256x128.

Resolution: 128x256x128

This simulation runs
at a low resolution of
 $96 \times 192 \times 96$.

Even at a lower resolution of
64x128x64, we observe
turbulent pyroclastic structures.

The vortex core structure remains virtually unchanged compared to the first frame even after 500 seconds of simulation.



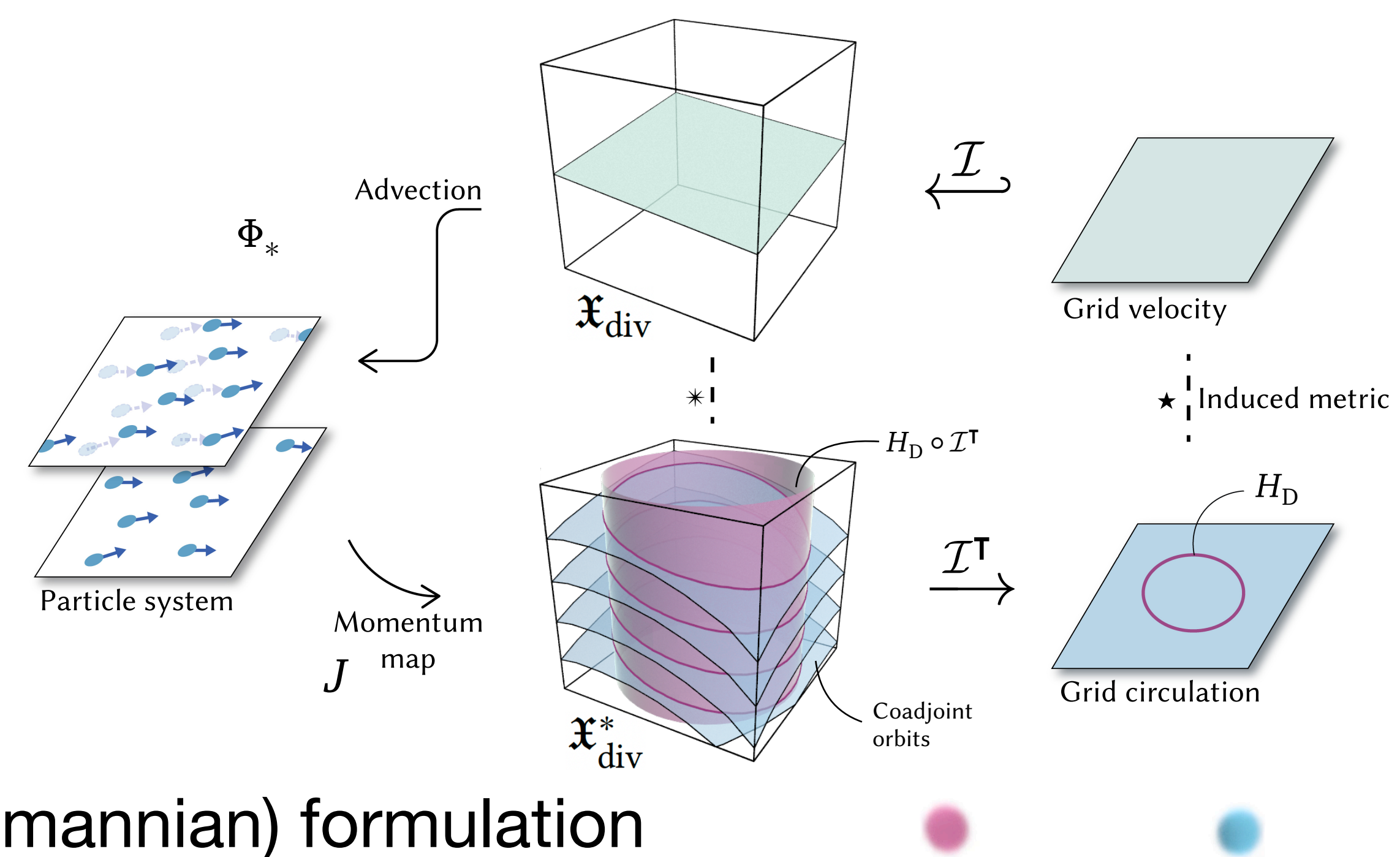
sped up playback x12.5



Conclusion

And Next Steps...

- Structures (e.g. symmetries) are essential
- Discretization benefits from a variational non-holonomic constrained (vakonomic/sub-Riemannian) formulation
- Similar to geometry processing, physical simulations benefit from structure-preservation



Thank you! Questions?

*Sina is going to be on the faculty
job market this coming cycle!*

*Hesper is going to be on the postdoc
job market this coming cycle!*